Recursive Graph Deduction and Reachability Queries

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Abstract - *In this paper, we discuss an adjustable strategy for the transitive closure compression to support graph reachability queries, asking whether a given node u in a directed graph G is reachable from another node v through a path. The main idea behind it is to define a series of graph deductions $G_0(V_0, E_0)$ (= G), $G_1(V_1, E_1)$, ..., $G_k(V_k, E_k)$ with $n_i > n_{i+1}$ (i = 0, ..., k - 1), where $n_i = |V_i|$. Each node v will be associated with an interval sequence $[\alpha_0^v, \beta_0^v)$, ..., $[\alpha_j^v, \beta_j^v)$ ($j \le k$ - 1) with each $[\alpha_i^v, \beta_i^v)$ used to check reachability in G_i. Together with a subgraph structure (called the core graph of G) and a small matrix (called the core matrix), the reachability checking can be done in O(k) time. The size of the compressed transitive closure is bounded by $O(kn_0 + b_k n_k)$, where b_k is the width of G_k, defined to be the size of a largest node subset U of G_k such that for every pair of nodes u, v ∈ U, there does not exist a path from u to v or from v to u in G_k. The time for generating such a compressed transitive closure is bounded by $O(e_0 + kn_0 + n_k^2 + b_k n_k \sqrt{b_k})$, where $e_0 = |E_0|$. For different applications, we can adjust k to different constants to get effective space reduction, but still have constant query time.*

Keywords: recursive graph deduction, transitive closure, reachability queries

1 Introduction

Given two nodes *u* and *v* in a directed graph *G*(*V*, *E*), we want to know if there is path from *u* to *v*. The problem is known as *graph reachability*. In many applications, such as recursive queries [7, 12, 26, 27], object-oriented databases [15], as well as XML query processing, transportation network, internet traffic analyzing, semantic web, computer vision, and metabolic network [28], graph reachability is one of the most basic operations, and therefore needs to be efficiently supported. Among them, some use sparse graphs, such as XML documents which are a labeled tree plus several *IDREF/ID* links, and metabolic networks which are an evolution tree plus some genes' interactions.

A naive method is to precompute the reachability between every pair of nodes – in other words, to compute and store the transitive closure (*TC* for short) of a graph. Then, a reachability query can be answered in constant time. However, this requires $O(|V|^2)$ space, which makes it impractical for massive graphs. Another method is to compute the shortest path from *u* to *v* over such a large graph on demand, which results in high query processing cost.

In this paper, we propose a new method to compress transitive closures to support reachability queries. The main idea behind it is to find a series of graph deductions $G_0(V_0, E_0)$ (= G), $G_1(V_1, E_1)$, ..., $G_k(V_k, E_k)$ with $n_i > n_{i+1}$ ($n_i = |V_i|$, i = 0, ..., k - 1), and associate each node v with an interval sequence $[\alpha_0^v, \beta_0^v)$, ..., $[\alpha_j^v, \beta_j^v)$ ($j \le k$ - 1) with each $[\alpha_i^v, \beta_i^v)$ used to check reachability in G_i. In addition, a subgraph G_{core} (called the *core graph* of G) and a small matrix M_{core} (called the *core matrix* of G) are constructed. To check whether node u is reachable from node v through a path in G, we will search two paths in G_{core}:

$$v_0 = v \to v_1 \to ... \to v_j \qquad (0 \le j \le k)$$
$$u_0 = u \to u_1 \to ... \to u_j.$$

Along these two paths, we first check whether $\alpha_0^{u_0} \in [\alpha_0^{v_0}, \beta_0^{v_0})$. If it is the case, u must be a descendant of v. Otherwise, we traverse to v_1 and u_1, respectively; and check $\alpha_1^{u_1}$ against $[\alpha_1^{v_1}, \beta_1^{v_1})$. We repeat this process until we meet v_j and u_j with one of the following conditions satisfied:

1. $j < k$ but we cannot further traverse from v_j or from u_j to a next node.
2. $j = k$.

In case (1), we report that u is reachable from v if $\alpha_j^{u_j} \in [\alpha_j^{v_j}, \beta_j^{v_j})$. Otherwise, u cannot be a descendant of v. In case (2), we need to check M_{core} to know whether u_k is reachable from v_k. The reachability of u_k from v_k implies the reachability of u from v.

Obviously, the above process needs only O(k) time to check reachability. But the space overhead is dramatically reduced to $O(\sum_{j=0}^{k-1} n_i + b_k n_k) \le O(kn_0 + b_k n_k)$, where b_k is the width of G_k, defined to be the size of a largest node subset U of G_k such that for every pair of nodes u, $v \in U$, there does not exist a path from u to v or from v to u in G_k. It is a biased trade-off of time for space. While the query time increases linearly, the space overhead decreases quadratically (in the sense that both b_k and n_k are reduced at each step of graph deduction.) The time for generating a compressed transitive closure is bounded by $O(e_0 + \sum_{j=0}^{k-1} n_i + n_k^2 + b_k n_k \sqrt{b_k}) \le O(e_0 + kn_0 + n_k^2 + b_k n_k \sqrt{b_k})$, where $e_0 = |E_0|$. More importantly, our

method provides a flexible strategy to compress transitive closures. For different applications, we can set k to be different constants to get effective space deduction, but with constant query time.

The rest of the paper is organized as follows. In Section 2, we review some related work. In Section 3, we show a basic method for the *TC* compression based on a kind of graph decomposition, which will be used in our strategy. In Section 4, we discuss our first graph deduction. Section 5 is devoted to the recursive graph deduction. The paper concludes in Section 6.

2 Related work

In the past two decades, many interesting labeling-based methods have been proposed to speed up the reachability query evaluation, which can be roughly classified into two groups: strategies for sparse graphs and strategies for non-sparse graphs. In the following, some of them are reviewed.

- *Strategies for sparse graphs*

In [19], Wang et al. discussed an interesting approach, called *Dual-I*, for sparse graphs $G(V, E)$. Its space overhead is bounded by $O(n + t^2)$ and can be produced in $O(n + e + t^3)$ time, where t is the number of non-tree edges. The query time is O(1). As a variant of *Dual-I*, one can also store a matrix N (maintained by *Dual-I*) as a tree (called a *TLC* search tree), which can reduce the space overhead from a practical viewpoint, but increases the query time to log$t$. This scheme is referred to as *Dual-II*.

The method proposed by Cohen *et al.* [6] labels a graph based on the so-called 2-*hop covers*. It is also designed specifically for sparse graphs. A hop is a pair (h, v), where h is a path in G and v is one of the endpoints of h. A 2-hop cover is a collection of hops H such that if there are some paths from v to u, there must exist $(h_1, v) \in H$ and $(h_2, u) \in H$ and one of the paths between v and u is the concatenation h_1h_2. The main theoretical barrier of this method is that finding a 2-hop cover of minimum size is an *NP*-hard problem.

- *Strategies for non-sparse graphs*

In [13], Jagadish suggested an interesting method to decompose a *DAG* (*directed acyclic graph*) into node-disjoint chains. Then, each node v is assigned an index (i, j), where i is a chain number, on which v appears, and j indicates v's position on the chain. These indexes can be used to check reachability efficiently. For this method, the space overhead and the query time are bounded by $O(\kappa n)$ and O(1), respectively, where κ is the number of chains. However, to find a minimized set of chains for a graph, Jagadish's algorithm needs $O(n^3)$ time (see page 566 in [13]). The method discussed in [8, 9] greatly improves Jagadish's method. It uses only $O(n^2 + bn)$ time to decompose a DAG into a minimum set of node-disjoint chains, where b represents G's width. Its space overhead is $O(bn)$. In [9], the concept of the so-called general spanning tree is introduced, in which each edge corresponds to a path in G. Based on this, the real space requirement becomes smaller than $O(bn)$, but the query time increases to log$b$.

In [1], Agrawal *et al*. proposed a method based on interval labeling. As with the dual labeling, this method first figures out a spanning tree T and assign to each node v in T an interval (a, b), where b is v's postorder number (which reflects v's relative position in a postorder traversal of T); and a is the smallest postorder number among v and v's descendants with respect to T (i.e., all the nodes in $T[v]$, the subtree rooted at v). Another node u labeled (a', b') is a descendant of v (with respect to T) iff $a \le b' < b$. This idea originates from Schubert e*t al*. [21]. In a next step, each node v in G will be assigned a sequence $L(v)$ of intervals such that another node u in G with interval (x, y) is a descendant of v (with respect to G) iff there exists an interval (a, b) in $L(v)$ such that $a \le y < b$. The length of a sequence associated with a node in G is bounded by $O(\kappa')$, where κ' is the number of the leaf nodes in T. So the time and space complexities are bounded by $O(\kappa' e)$ and $O(\kappa' n)$, respectively. The querying time is bounded by $O(\log\kappa')$. In the worst case, $\kappa' = O(n)$.

3 About TC compression

In this section, we briefly show the method discussed in [8, 9] to compress *TC* based on a graph decomposition, by which G is not deducted, but directly decomposed into a minimum set of node-disjoint chains. On a chain, if node v is above node u, then there is a path from v to u in G. Without loss of generality, we assume that G is *acyclic* (i.e., G is a DAG.) If not, we will find all the *strongly connected components* (*SCCs*) of G and collapse each of them into a representative node. Obviously, each node in an *SCC* is equivalent to its representative node as far as reachability is concerned. This process takes $O(e)$ time using Tarjan's algorithm [18].

For illustration, consider the DAG shown in Fig. 1(a). Its transitive closure (stored as a 0-1 matrix) is shown in Fig. 1(b). Obviously, it requires $O(n^2)$ space.

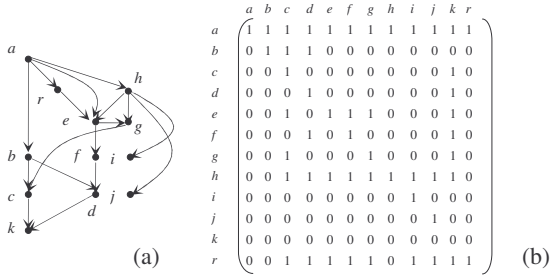


Fig. 1. A DAG and a matrix representing its transitive closure

Using the algorithm discussed in [8, 9], however, we can always decompose a DAG into a minimum set of node-disjoint chains, as illustrated in Fig. 2(a), in which the 1st, 2nd and 3rd chains are in fact three paths. But the 4th is a non-trivial chain (i.e., it is not a path since node a is connected to node i through a path of length 2, instead of an edge.) The fifth chain contains only a single node. We also remark that the width of the graph is 5 since there exists a subset $U = \{b, f, g, i, j\}$, in which each pair of nodes are not connected. So it

is not possible to decompose the graph into a set with fewer chains.

We can then assign an index to each node in G as follows:

(1) Number each chain and number each node on a chain.
(2) The jth node v on the ith chain will be assigned a pair (i, j) as its index, denoted *index*(v).

In addition, each node v on the ith chain will be associated with an index sequence of length b - 1: $(1, j_1) \ldots (i - 1, j_{i-1})$ $(i + 1, j_{i+1}) \ldots (b, j_b)$ (as illustrated in Fig. 2(a)) such that any node with index (x, y) is a descendant of v if $x = i$ and $y \geq j$ or $x \neq i$ but $y \geq j_x$, where b is the number of the node-disjoint chains, equal to G's width [8]. (Here, a node is considered to be an ancestor of itself.)

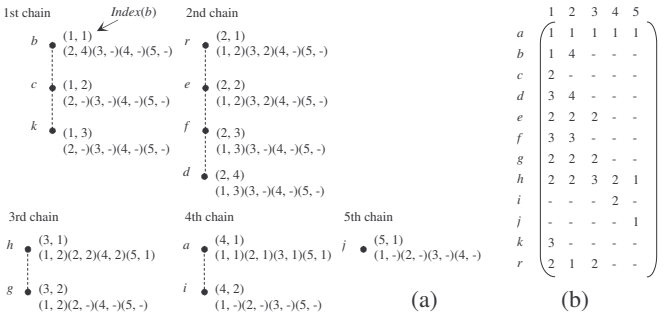


Fig. 2. A set of chains and a matrix

We can also store all the index sequences as an $n \times b$ matrix M_G as shown in Fig. 2(b), in which each entry $M_G(v, j)$ is the jth element in the index sequence associated with node v. So, a node u with *index*(u): (i, j) is a descendant of another node v iff $M_G(v, i) \leq j$. Thus, based on M_G, a reachability checking needs only O(1) time. M_G is called the *reachability matrix* of G.

Obviously, M_G dominates the space requirement. If we want to further decrease the space, M_G should be reduced in some way, but without sacrificing too much query time. In the the next section, we address this issue in great detail.

4 Graph deduction

In this section, we discuss our graph deduction. First, we give the general definition of graph deduction in 4.1. Then, in 4.2, we discuss a very important concept, the so-called *critical nodes*, based on which a special kind of graph deduction can be established for checking reachability. In 4.3, we show how the reachability can be checked based on such a deducted graph.

4.1 Basic definitions

Let G be a directed graph. We use $V(G)$ and $E(G)$ to represent its node set and edge set, respectively. It is well known that the preorder traversal of G introduces a spanning tree (forest) T. With respect to T, $E(G)$ can be classified into four groups:

- *tree edges* (E_{tree}): edges appearing in T.
- *cross edges* (E_{cross}): any edge (u, v) such that u and v are not on a path in T.
- *forward edges* ($E_{forward}$): any edge (u, v) not appearing T, but there exists a path from u to v in T.
- *back edge* (E_{back}): any edge (u, v) not appearing T, but there exists a path from v to u in T.

All cross, forward, and back edges are referred to as non-tree edges. If G is a DAG, we do not have back edges since a back edge implies a cycle.

For illustration, consider the DAG shown in Fig. 1(a) once again. For it, we may find a spanning tree as shown in Fig. 3, in which each solid arrow stands for a tree edge while a dashed arrow for a non-tree edge.

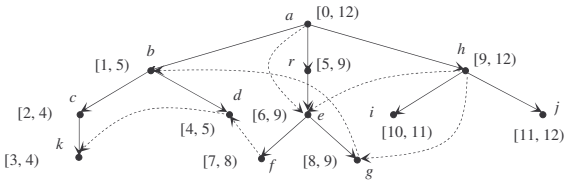


Fig. 3. A spanning tree and intervals

As in [19], we can assign each node v in T an interval [α, β), where α is v's preorder number (denoted *pre*(v)) and β - 1 is equal to the largest preorder number among all the nodes in $T[v]$. So another node u labeled [α', β') is a descendant of v (with respect to T) iff α' ∈ [α, β) [19], as illustrated in Fig. 3. If α' ∈ [α, β), we say, [α', β') is subsumed by [α, β).

Let T be a spanning tree of G. Let $e = (u, v)$ be an edge. By $G\backslash e$ we denote a graph obtained from G by performing one of the following two operations:

1. If e is a forward edge, remove e from G.
2. If e is a tree edge (an edge in T), remove e and v; and any edge incident to v becomes incident to u.

Especially, for a tree edge e, $T\backslash e$ is still a spanning tree of $G\backslash e$, based on which we define the following concept.

Definition 1 (*graph deduction*) A graph G_1 is a deduction of G if there are graphs $G^{(0)}$, ..., $G^{(j)}$ and forward or tree edges $e_i \in G^{(i)}$ such that $G^{(0)} = G$, $G^{(j)} = G_1$, and $G^{(i+1)} = G^{(i)}\backslash e_i$.

The graph deduction is a special kind of graph minors [10]. Our purpose is to find a graph deduction G_1 of G such that

i) the reachability matrix of G_1, M_{G_1}, is smaller than M_G;
ii) the reachability of G can be checked with help of M_{G_1};
iii) the query time is increased only by a constant.

In the following, we discuss how such a G_1 can be found.

4.2 Critical nodes and critical subgraphs

Our main idea is to recognize a subset of nodes, the so-called *critical nodes*, which enable us to construct a graph deduction satisfying all the above properties.

We denote by E' the set of all cross edges. Denote by V' the set of all the *end points* of the cross edges. That is, V' =

$V_{start} \cup V_{end}$, where V_{start} contains all the *start nodes* while V_{end} all the *end nodes* of the cross edges. In Fig. 4(a), we show the corresponding V_{start} and V_{end} for the graph shown in Fig. 3. No attention is paid to the forward edge (a, e) in the graph since it can simply be removed as far as the reachability is concerned.

Definition 2 (*anti-subsuming subset*) A subset $S \subseteq V_{start}$ is called an *anti-subsuming set* iff $|S| > 1$ and no two nodes in S are related by an ancestor-descendant relationship with respect to T.

As an example, consider the spanning tree shown by the solid arrows in Fig. 3. With respect to this spanning tree, we have altogether 11 anti-subsuming subsets as shown in Fig. 4(b).

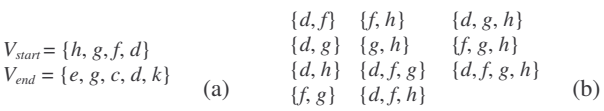


Fig. 4. Anti-subsuming sbsets

Definition 3 (*critical nodes*) A node v in a spanning tree T of G is *critical* if $v \in V_{start}$ or there exists an anti-subsuming subset $S = \{v_1, v_2, ..., v_k\}$ for $k \geq 2$ such that v is the lowest common ancestor of $v_1, v_2, ..., v_k$.

For example, in the spanning tree shown in Fig. 3, node e is the lowest common ancestor of $\{f, g\}$, and node a is the lowest common ancestor of $\{d, f, g, h\}$. So e and a are two critical nodes. In addition, each $v \in V_{start}$ is a critical node. So, all the critical nodes of G with respect to T are $\{d, f, g, h, e, a\}$. We call a critical node *trivial* if it belongs to V_{start}; otherwise, *non-trivial*. We denote by V_c all the critical nodes. Based on the concept of critical nodes, we can now define our graph deduction G_1, called a *critical subgraph* of G.

Let T be a spanning tree of G. Denote by T_r a reduction of T obtained by removing all those nodes $v \notin V_c \cup V_{end}$. Deleting a node v entails connecting v's parent to each of v's children. So, removing a node in this way corresponds to the elimination of a tree edge.

For example, for the spanning tree T shown in Fig. 3, its T_r is a tree shown in Fig. 5. It is obtained by removing the nodes b, r, i, and j one by one. Note that none of them belongs to $V_c \cup V_{end}$. (Remember that $V_c = \{a, d, e, f, g, h\}$ and $V_{end} = \{c, d, e, g, k\}$.)

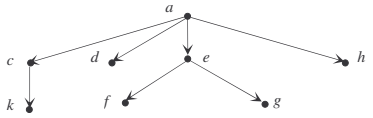


Fig. 5. A tree reduction

Definition 4 (*critical subgraph*) Let $G(V, E)$ be a DAG. Let T be a spanning tree of G. The critical subgraph G_c of G with respect to T is graph with node set $V(T_r)$ and edge set $E(T_r) \cup E_{cross}$.

In Fig. 6(a), we show a critical subgraph of the graph shown in Fig. 3, with respect to the corresponding spanning tree.

Again, using the algorithm discussed in [8, 9], we can decompose it into a set of node-disjoint chains as illustrated in Fig. 6(b); and construct a reachability matrix as shown in Fig. 6(c), which is much smaller that the matrix shown in Fig. 1(b), and even smaller than that shown in Fig. 2(b). However, it seems that such a matrix can only be used to check the reachability between the nodes in G_c.

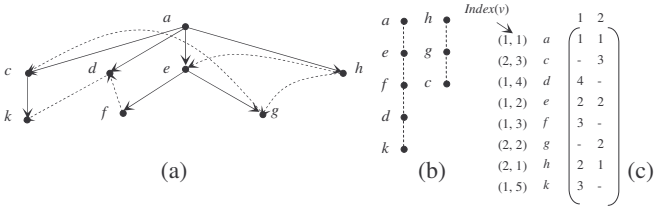


Fig. 6. A G_c, its decomposition and its reachability matrix

The question is: can such a matrix also be used to check the reachability between the nodes in G? In the next section, we answer this question.

4.3 Evaluation of reachability queries

For any two node u, v appearing on a path in T, their reachability can be checked using their associated intervals. However, if they are not on the same path, we have to use G's reachability matrix M_G to check whether u is reachable from v through a path in G, or *vice versa*.

Now what we have is M_{G_c}, rather than M_G. How can we check the reachability from v to u in G?

To do this, we need another concept, the so-called *anchor nodes*.

Definition 5 (*anchor nodes*) Let G be a DAG and T a spanning tree of G. Let v be a node in T. Denote by C_v all the critical nodes in $T[v]$. We associate two anchor nodes with v as below.

i) A node $u \in C_v$ is called an anchor node (of the first kind) of v if u is closest to v. u is denoted v^*.

ii) A node w is called an anchor node (of the second kind) of v if it is the lowest ancestor of v (in T), which has a cross incoming edge. w is denoted v^{**}.

For example, in the graph shown in Fig. 3, $r^* = e$. It is because node e is critical and closest to node r in $T[r]$. But r^{**} does not exist since it does not have an ancestor which has a cross incoming edge. In the same way, we find that $e^* = e^{**} = e$. That is, both the first and second kinds of anchor nodes of e are e itself. We can easily recognize the anchor nodes for all the other nodes in the graph.

The following two lemmas are critical to our non-tree labeling method.

Lemma 1 Any critical node in C_v appears in $T[v^*]$.

Proof. Assume that there exists a critical node u in C_v, which does not appear in $T[v^*]$. Let $u_1, ..., u_k$ be all the critical nodes in $T[v^*]$. Consider the lowest common ancestor node of u, u_1, ..., u_k. It must be an ancestor node of v^*, which is closer to v than v^*, contradicting the fact that v^* is the closest critical node (in $T[v]$) to v.

Lemma 2 Let u be a node, which is not an ancestor of v in T; but v is reachable from u via some non-tree edges. Then, any way for u to reach v must be through v**.

Proof. This can be seen from the fact that any node which reaches v via some cross edges is through v** to reach v.

Definition 6 (*non-tree labels*) Let v be a node in G. The non-tree label of v is a pair $<x, y>$, where

- $x = v$* if v* exists. If v* does not exists, let x be the special symbol "-".
- $y = v$** if v** exists. If v** does not exist, let y be "-".

The purpose of the non-tree labeling is to use the anchor nodes u** and v* to check the reachability of u from v through cross edges.

Example 1 Consider G and T shown in Fig. 3. The non-tree labels of the nodes are shown in Fig. 7.

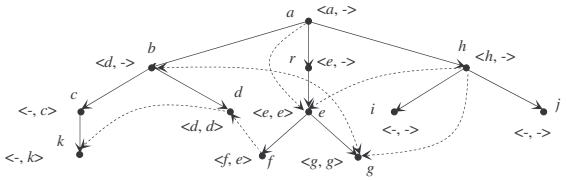


Fig. 7. Non-tree labels

In this figure, we can see that the non-tree label of node r is $<e, ->$ because (1) r* = e; and (2) r** does not exist. Similarly, the non-tree label of node f is $<f, e>$. It is because f* is f itself; but f** is e (see Fig. 7).

Especially, we notice that node r and node d are not on the same path in T. But d is a descendant of r. Such a reachability has to be checked by using their anchor nodes. In fact, we have d** = d, $index(d) = (1, 4)$, r* = e, and $(e, 1) = 2 < 4$ (see Fig. 6(c)), which shows that d** is a descendant of r*. By the following proposition, this indicates that d is a descendant of r.

Proposition 3 Assume that u and v are two nodes in G, labeled $([\alpha_1, \beta_1), <x_1, y_1>)$ and $([\alpha_2, \beta_2), <x_2, y_2>)$, respectively. Here, $[\alpha_i, \beta_i)$ $(i = 1, 2)$ is the tree label while $<x_i, y_i>$ $(i = 1, 2)$ is the non-tree label. Node u is reachable from v iff one of the following conditions holds:

(i) $[\alpha_1, \beta_1)$ is subsumed by $[\alpha_2, \beta_2)$ (i.e., $\alpha_1 \in [\alpha_2, \beta_2)$), or

(ii) $index(y_1) = (x, y)$, and $(x_2, x) \le y$.

Proof. The proposition can be derived from the following two facts:

(1) u is reachable from v through tree edges iff $[\alpha_1, \beta_2)$ is subsumed by $[\alpha_2, \beta_2)$.

(2) In terms of Lemma 2, u is reachable from v via non-tree edges iff u** exists and its index $index(u$**$)$ is a pair (x, y) such that $(v$*$, x) \le y$.

Now we analyze the time complexity of the whole process. It mainly comprises three parts:

i) the cost to find a spanning tree of $G(V, E)$;

ii) the cost to find all the critical nodes, and establish G_c; and

iii) the cost to decompose the critical graph of G into a minimum set of node-disjoint chains.

The cost of the first part is $O(|V| + |E|)$. The second part is also bounded by $O(|V| + |E|)$. According to [8, 9], the third part is bounded by $O(m^2 + bm\sqrt{b})$, where $m = |V(G_c)|$, and b is the width of G_c.

Proposition 4 Let G be a DAG with n nodes and e edges. Then, the labeling time of our method is bounded by $O(n + e + m^2 + bm\sqrt{b})$.

Proof. See the above analysis.

5 Recursive graph deduction

Now we discuss the recursive graph deduction.

5.1 Recursive deduction

In the previous section, we showed how to reduce G to G_c. Especially, using T and M_{G_c}, we can check the reachability of G in constant time.

However, it can be observed that G_c itself can be further reduced, leading to a further decrement of space requirement. In fact, using the method discussed in 4.1 and 4.2, we can find a series of graph deductions:

$$G_0 = G, G_1, ..., G_k, \qquad (k \ge 1)$$

where G_i is a critical subgraph of G_{i-1} $(i = 1, ..., k)$.

In order to construct such critical subgraphs, a series of spanning trees have to be established:

$$T_0, T_1, ..., T_{k-1},$$

where each T_i is a spanning tree of G_i $(i = 0, ..., k - 1)$, used to construct G_{i+1}.

Finally, similar to M_{G_k}, we will have a reachability matrix for G_k, called the *core matrix* of G and denoted M_{core}. To check reachability efficiently, each node v in G will be associated with two sequences: an interval sequence and an anchor node sequence:

1) $[\alpha_0^v, \beta_0^v), ..., [\alpha_j^v, \beta_j^v) \qquad (j \le k - 1)$

where each $[\alpha_i^v, \beta_i^v)$ is an interval generated by labeling T_i;

2) $(x_0^v, y_0^v), ..., (x_j^v, y_j^v),$

where each x_i^v is a pointer to an anchor node of the first kind (a node appearing in G_{i+1}) while each y_i^v a pointer to an anchor node of the second kind (also, a node in G_{i+1}).

Fig. 8(a) helps for illustration.

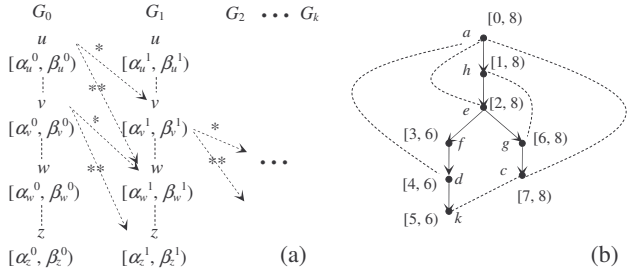


Fig. 8. Illustration for anchor nodes and a spanning tree

In this figure, a dashed arrow marked with * stands for a pointer to an anchor node of the first kind while a dashed arrow marked with ** for a pointer to an anchor node of the

second kind. For each node appearing in G_i, an interval and a pair of anchor pointers will be created. So, if node v appears in $G_0 = G$, G_1, ..., G_j for some $j \le k$, it will be associated with two sequence of length j, as described above.

Example 3 Denote by G_0 and G_1 the graph shown in Fig. 3 and the critical graph shown in Fig. 5(a), respectively. Fig. 8(b) shows a possible spanning tree T_1 of G_1, and the corresponding tree labels.

With respect to T_1, we do not have any anti-subsuming subset (see Fig. 9(a)). Thus, we have no non-trivial critical nodes. The critical graph G_2 is shown in Fig. 9(b), containing only one edge (c, k). Tackling this edge as a chain (as shown in Fig. 9(c)), we generate a reachability matrix as shown in Fig. 9(d). The non-tree labels of the nodes in G_1 are shown in Fig. 10(a). The interval sequence and the anchor node sequence for each node of G are shown in Fig. 10(b).

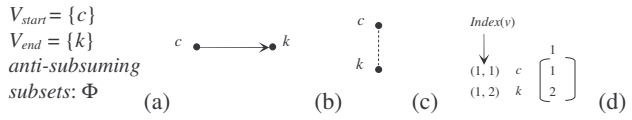
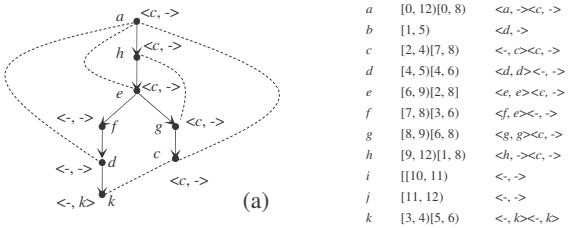


Fig. 9. Anti-subsuming subsets, and reachability matrix



a	[0, 12)[0, 8)	<a, -><c, ->
b	[1, 5)	<d, ->
c	[2, 4)[7, 8)	<-, c><c, ->
d	[4, 5)[4, 6)	<d, d><-, ->
e	[6, 9)[2, 8]	<e, e><c, ->
f	[7, 8)[3, 6)	<f, e><-, ->
g	[8, 9)[6, 8)	<g, g><c, ->
h	[9, 12)[1, 8)	<h, -><c, ->
i	[[10, 11)	<-, ->
j	[11, 12)	<-, ->
k	[3, 4)[5, 6)	<-, k><-, k>
r	[5, 9)	<e, ->

Fig. 10. Non-tree labels and sequences associated with nodes

5.2 Evaluation of reachability queries

Now we discuss how to use the interval sequences and anchor node sequences to check reachability. First, we notice that the anchor node sequences imply a graph, called a *core graph* of G and denoted G_{core}, in which there exists an edge (u, v) iff there is an entry <x, y> in the anchor node sequence associated with u such that $x = v$, or $y = v$. The edge is labeled with {i, *} or {i, **}, depending on whether $x = v$, or $y = v$, where i indicates that <x, y> is the ith entry in the anchor node sequence. In Fig. 11, we show the core graph corresponding to the anchor node sequences shown in Fig. 10(b).

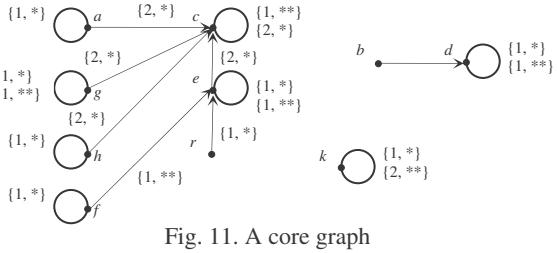


Fig. 11. A core graph

In the graph, edge (r, e) labeled with {1, *} represents that e is an anchor node (of the first kind) of r, which appears in G_1 while edge (a, c) labeled with {2, **} represents that c is an anchor node (of the second kind) of a, appearing in G_2. An edge with multiple labels represents several edges with different labels. For example, the edge (e, e) (represented as a loop) labeled with {1, *}, and {1, **} stands for two edges with each going from e to e, but labeled differently.

Remark that each node v in G_{core} is associated with an interval sequence $[\alpha_0^v, \beta_0^v)$, ..., $[\alpha_x^v, \beta_x^v)$ for some $x \ge 0$. In order to check whether v is an ancestor of u, we will search two paths in G_{core}, starting from v and u, respectively. The path starting from v, denoted P_1, contains only the edges labeled with (i, *) while the path starting from u, denoted P_2, contains only the edges labeled with (i, **). Each time we reach two nodes v' and u' through two edges labeled respectively with (i, *) and (i, **), we will check whether $[\alpha_i^{v'}, \beta_i^{v'})$ subsumes $[\alpha_i^{u'}, \beta_i^{u'})$. If it is the case, v is an ancestor of u. Otherwise, we traverse along P_1 and P_2, reaching v'' and u'' through two edges labeled respectively with (i + 1, *) and (i + 1, **) and checking $[\alpha_{i+1}^{v''}, \beta_{i+1}^{v''})$ against $[\alpha_{i+1}^{u''}, \beta_{i+1}^{u''})$. We continue this process. After j steps for some j, we will meet two nodes v''' and u''' such that v''' does not have an out-going edge labeled with (j + 1, *) or u''' does not have an out-going edge labeled with (j + 1, **). If $[\alpha_j^{v'''}, \beta_j^{v'''})$ subsumes $[\alpha_j^{u'''}, \beta_j^{u'''})$, v is an ancestor of u. Otherwise, we will check whether $M_{core}(v''', w) \le z$, where $index(u''') = (w, z)$.

Example 4 Consider the graph shown in Fig. 3 once again. To check whether g is an ancestor of k, we will explore two paths in the graph shown in Fig. 11, starting from g and c, respectively. First, we check $[\alpha_0^g, \beta_0^g) = [8, 9)$ against $[\alpha_0^k, \beta_0^k) = [3, 4)$ and find that [8, 9) does not subsume [3, 4) (see Fig. 10(b)). Then, we go from g along an edge labeled with (1, *) to g itself; and from k along an edge labeled with (1, **) to k itself (see Fig. 11). Now, we check $[\alpha_1^g, \beta_1^g) = [6, 8)$ against $[\alpha_1^k, \beta_1^k) = [5, 6)$. Since [6, 8) does not subsume [5, 6), we will continue to explore the two paths along next two edges labeled with (2, *) and (2, **), respectively, reaching two nodes c and k (see Fig. 11). Note that $index(k) = (1, 2)$ (see Fig. 9(d)). Since $(c, 1) = 1 < 2$, g is an ancestor of k.

Proposition 4 Let G be a DAG, and $G_0 = G$, G_1, ..., G_k be a series of graph deduction as described above. Let u and v be two nodes in G. u is reachable from v through a path in G iff there exist two paths in G_{core}:

$$v_0 = v \rightarrow v_1 \rightarrow ... \rightarrow v_j \qquad (0 \le j \le k)$$
$$u_0 = u \rightarrow u_1 \rightarrow ... \rightarrow u_j$$

such that each (v_{i-1}, v_i) is labeled with (i, *), each (u_{i-1}, u_i) is labeled with (i, **), and one of the following two conditions is satisfied:

1. $j < k$, and u_j is reachable from v_j through a path in T_j; or
2. $j = k$, and u_j is reachable from v_j through a path in G_k.

Proof. if-part. We prove the if-part by induction on k.

Basis step. When $k = 0$, 1, the proof is trivial.

Induction hypothesis. Assume that when $k = l$ the if-part holds. We consider the case when $k = l + 1$. If $j \le l$, in terms

of the induction hypothesis, the if-part holds. Assume that $j = l + 1$. Since u_{l+1} is reachable from v_{l+1} through a path in G_{l+1}, u_l must be reachable from v_l in G_l by Lemma 1 and 2. (Note that v_{l+1} is an anchor node of the first kind of v_l and u_{l+1} is an anchor node of the second kind of u_l.) In terms of the induction hypothesis, u is reachable from v.

Only-if-part. If $u_0 = u$ is reachable from $v_0 = v$, there will be a path in T_0 from v_0 to u_0 or u_1 is reachable from v_1 in G_1. Similarly, u_1 is reachable from v_1 in G_1, there will be a path in T_1 from v_1 to u_1, or u_2 is reachable from v_2 in G_2. Repeating this argument, we will get the proof.

The above proposition shows that to check whether u is reachable from v, we need to search two paths in G_{core} and at each step to examine whether $\alpha_i^u \in [\alpha_i^v, \beta_i^v)$. If $i = k$, we have to check whether $M_{core}(v_j, x) \le y$, where $index(u_j) = (x, y)$. Clearly, it needs $O(k)$ time. The space complexity is easy to analyze. Since for each appearance of a node v in G_i an interval for v is created, the space overhead is bounded by $O(\sum_{j=0}^{k-1} n_i + b_k n_k)$, where n_i is the size of G_i and $b_k n_k$ is the size of M_{core}. According to [8, 9], the decomposition of G_k into a minimum set of node-disjoint paths needs $O(n_k^2 + b_k n_k \sqrt{b_k})$ time. So, the total labeling time is $O(e_0 + \sum_{j=0}^{k-1} n_i + n_k^2 + b_k n_k \sqrt{b_k})$.

6 Conclusion

In this paper, a new method is proposed to compress transitive closures to support reachability queries. Its main idea behind it is to find a series of graph deductions $G_0(V_0, E_0)$ $(= G)$, $G_1(V_1, E_1)$, ..., $G_k(V_k, E_k)$ with $n_i > n_{i+1}$ $(n_i = |V_i|, i = 0, ..., k - 1)$, and associate each node v with an interval sequence $[\alpha_0^v, \beta_0^v)$, ..., $[\alpha_j^v, \beta_j^v)$ $(j \le k - 1)$ with each $[\alpha_i^v, \beta_i^v)$ used to check reachability in G_i. In addition, a subgraph G_{core} (called the *core graph* of G) and a small matrix M_{core} (called the *core matrix* of G) are constructed, based on which a reachability query can be answered in $O(k)$ time. But the space for storing a compressed transitive closure is only k times larger than the original graph. For different applications, k can be set to different constants.

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