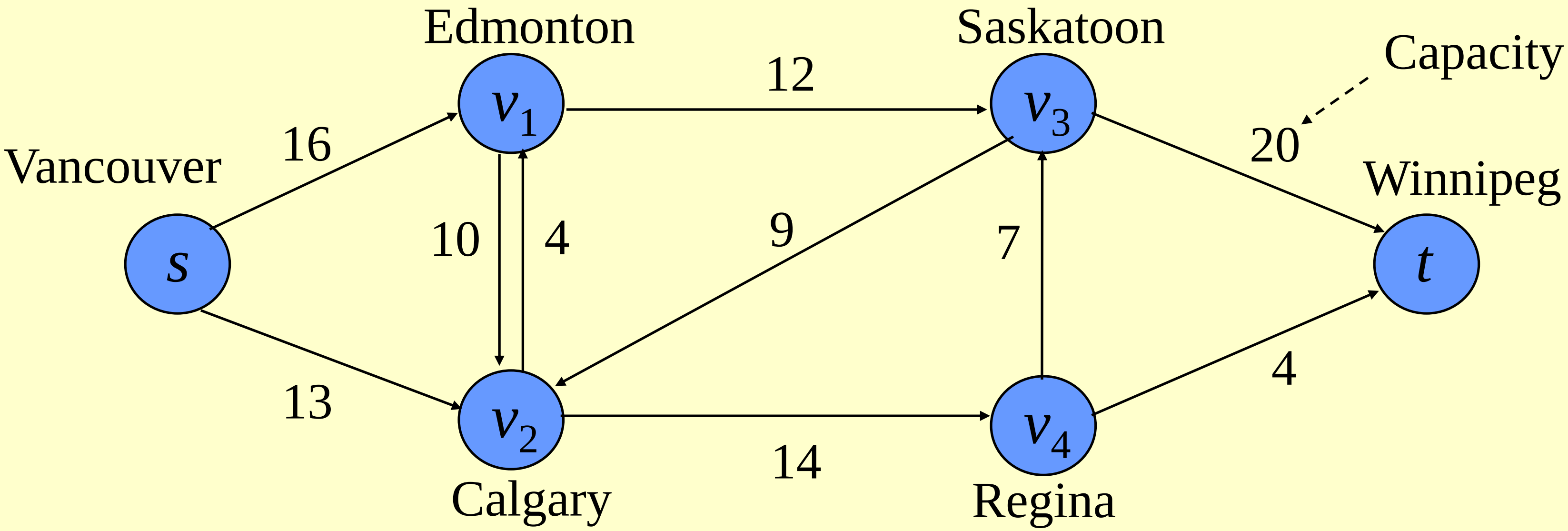
Flow Network

- Flow network and flows
- Ford-Fulkerson method to find a maximum flow
  - Residual networks
  - Augmenting paths
  - Cuts of flow networks
- Max-flow min-cut theorem

Chapter 26: Maximum Flow

- **A directed graph is interpreted as a flow network:**
  - **A material coursing through a system from a source, where the material is produced, to a sink, where it is consumed.**
  - **The source produces the material at some steady rate, and the sink consumes the material at the same rate.**
- **Maximum flow problem: to compute the greatest rate at which material can be shipped from the source to the sink.**

Example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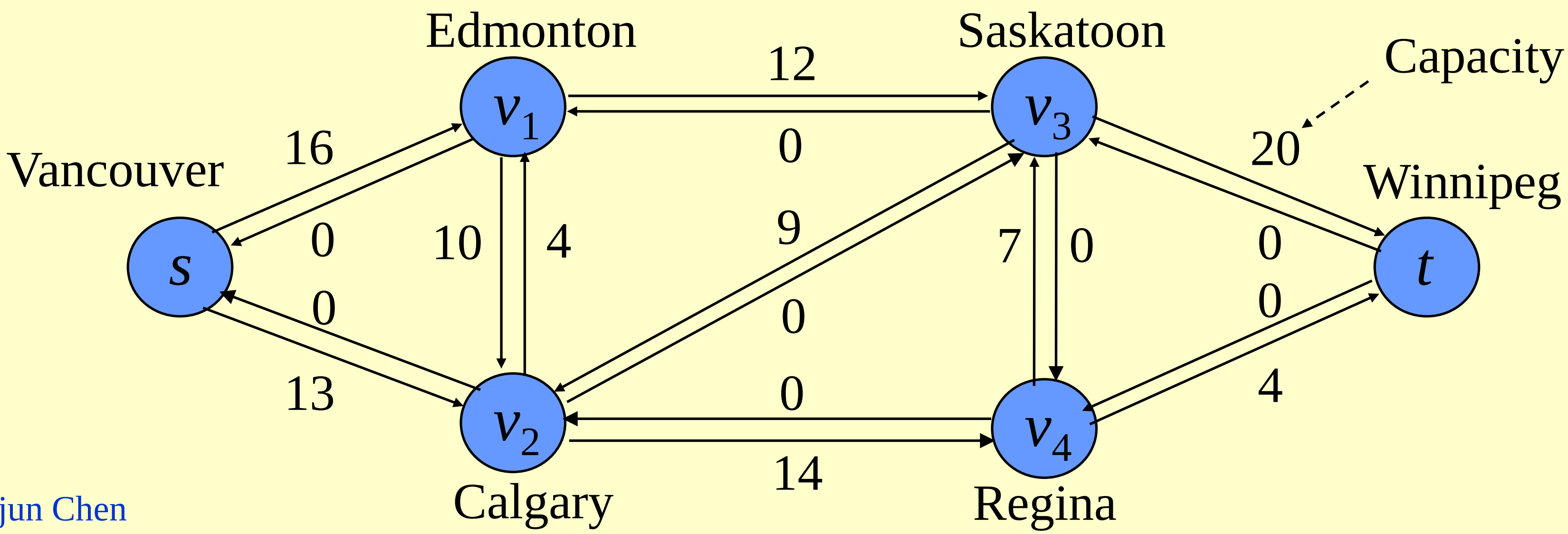
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- Applications which can be modeled by the maximum flow
  - Liquids flowing through pipes
  - Parts through assembly lines
  - current through electrical network
  - information through communication network

■ Definition – flow networks and flows

- A flow network $G = (V, E)$ is a directed graph in which each edge $(u, v) \in E$ has a nonnegative capacity $c(u, v) \geq 0$.
- source: s; sink: t
- For every vertex $v \in V$, there is a path:

  $s \rightsquigarrow v \rightsquigarrow t$

- A flow in G is a real-valued function f: $V \times V \rightarrow \mathbf{R}$ that satisfies the following properties:

  Capacity constraint: For all $u, v \in V$, $f(u, v) \leq c(u, v)$.

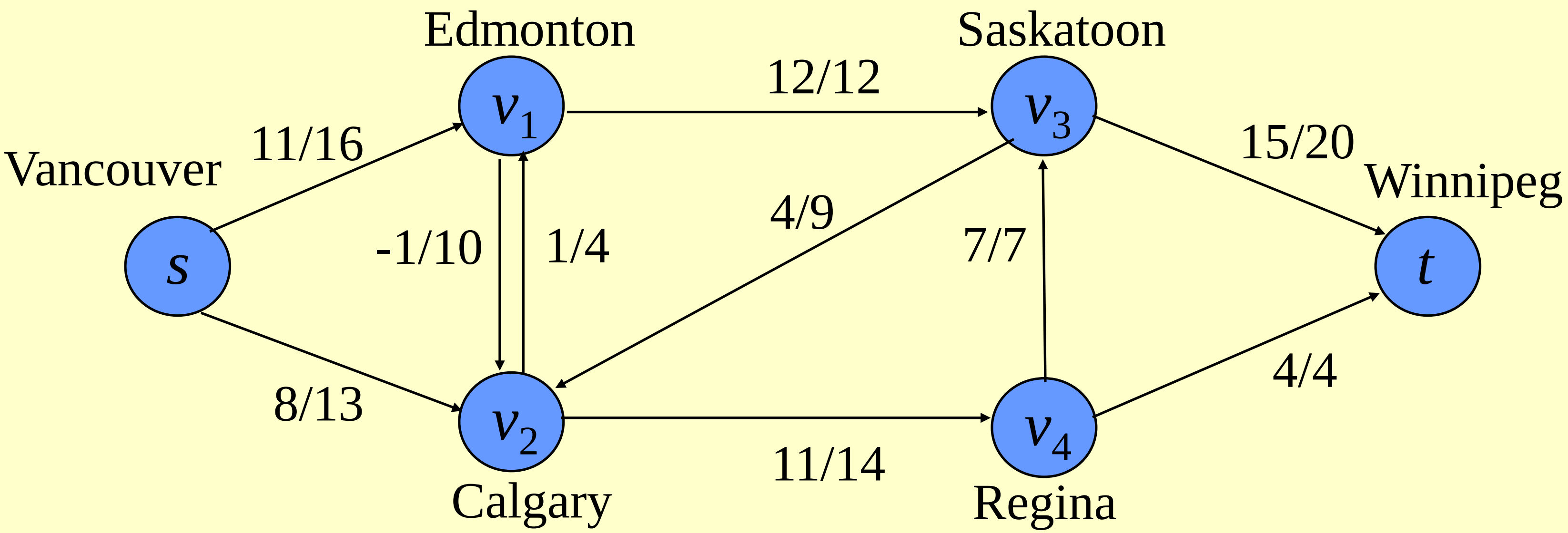
  Skew symmetry: For all $u, v \in V$, $f(u, v) = - f(v, u)$.

  Flow conservation: For all $u \in V - \{s, t\}$, $\sum_{v \in V} f(u, v) = 0$.

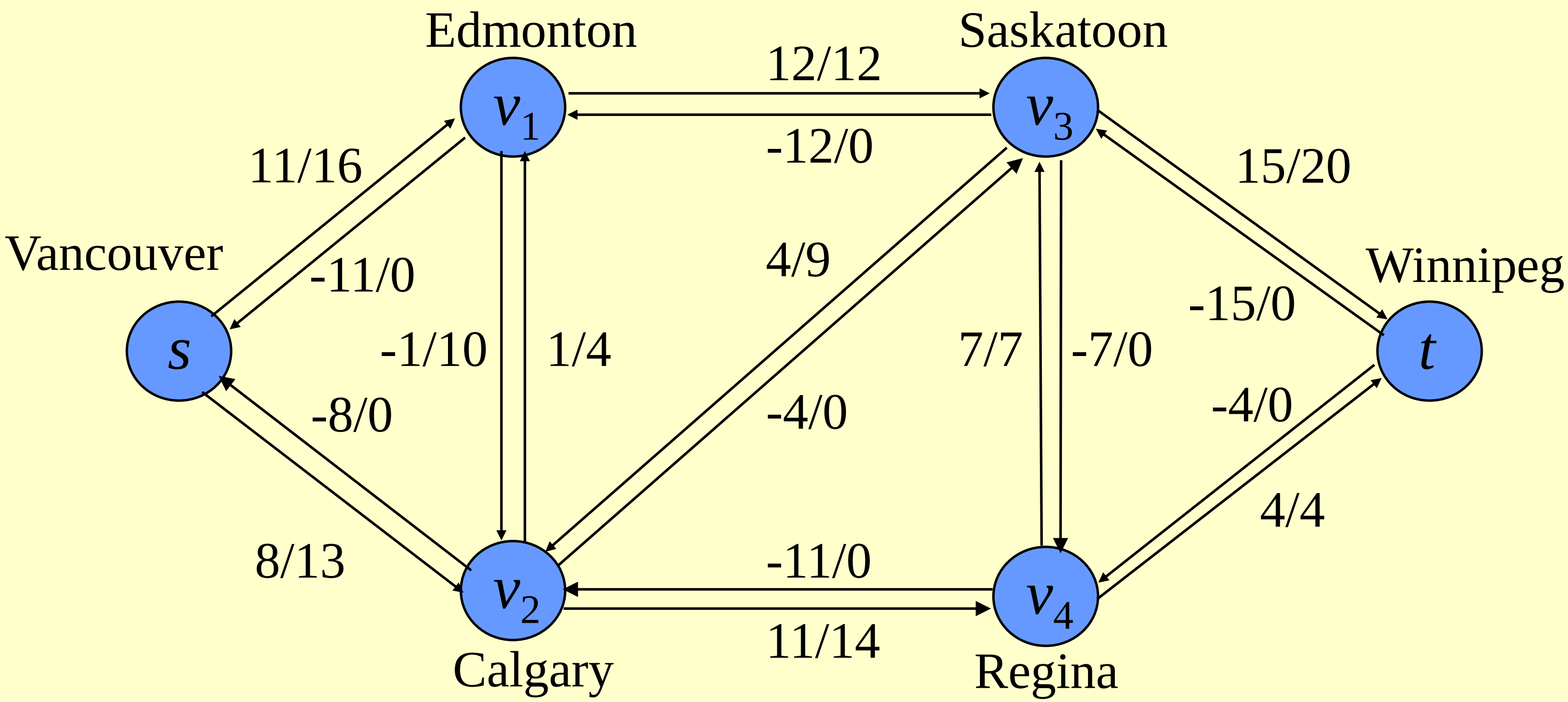
The quantity $f(u, v)$, which can be positive, zero, or negative, is called the ***flow*** from vertex u to vertex v. The value of a flow f is defined as the total flow out of the source

$$|f| = \sum_{v \in V} f(s, v)$$

- Example



■ Example



$\sum_{v \in V} f(u,v) = 0$. The total flow out of a vertex is 0.

$\sum_{u \in V} f(u,v) = 0$. The total flow into a vertex is 0.

The *total positive flow* entering a vertex *v* is defined by

$$\sum_{u \in V,\, f(u,v)>0} f(u,v)$$

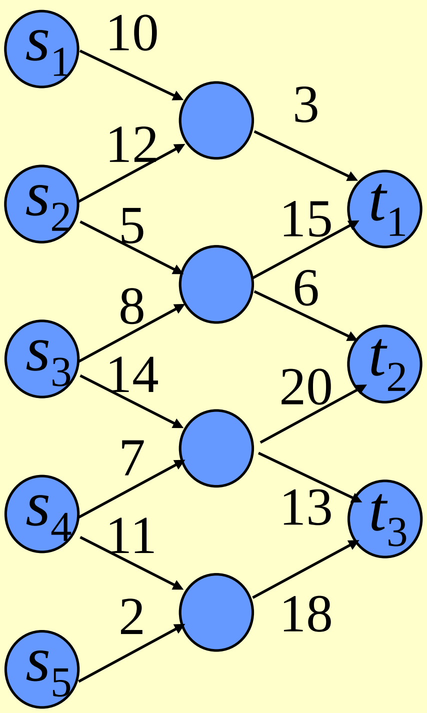
The *total net flow* at a vertex is the total positive flow leaving the vertex minus the total positive flow entering the vertex.

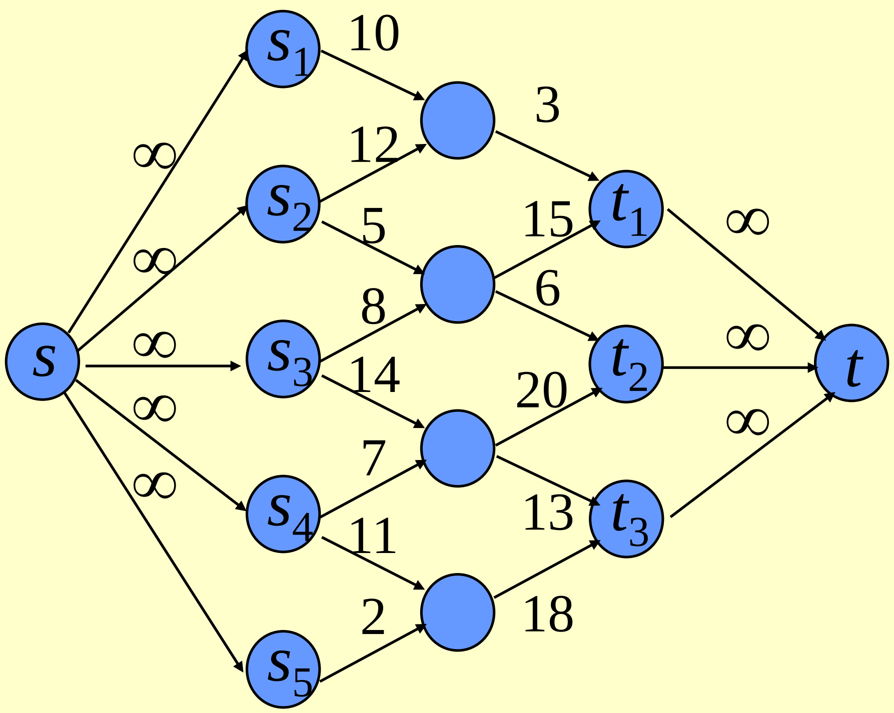
The *interpretation* of the flow-conservation property:

- The total positive flowing entering a vertex other than the source or sink must equal the total positive flow leaving that vertex.
- For all $u \in V - \{s, t\}$, $\sum_{v \in V} f(u,v) = 0$. That is, the total flow out of *u* is 0.
  For all $v \in V - \{s, t\}$, $\sum_{u \in V} f(u,v) = 0$. That is, the total flow into *v* is 0.

■Networks with multiple sources and sinks

- Introduce *supersource* *s* and *supersink* *t*





■ Working with flows

- implicit summation notation

  $$f(X, Y) = \sum_{x \in X} \sum_{y \in Y} f(x, y)$$

  The flow-conservation constraint can be re-expressed as

  $f(u, V) = 0$ for all $u \in V - \{s, t\}$.

- **Lemma 26.1** Let $G = (V, E)$ be a flow network, and let f be a flow in G. Then, the following equalities hold:

  1. For all $X \subseteq V$, we have $f(X, X) = 0$.
  2. For all $X, Y \subseteq V$, we have $f(X, Y) = - f(Y, X)$.
  3. For all $X, Y, Z \subseteq V$ with $X \cap Y = \emptyset$, we have the sums

  $$f(X \cup Y, Z) = f(X, Z) + f(Y, Z),$$
  $$f(Z, X \cup Y) = f(Z, X) + f(Z, Y).$$

1. For all $X \subseteq V$, we have $f(X, X) = 0$.

$X = \{x_1, \ldots, x_n\}$

$f(X, X) = \sum_{x \in X} \sum_{y \in X} f(x, y)$

$= f(x_1, x_2) + f(x_1, x_3) + \ldots + f(x_1, x_n) +$

$f(x_2, x_1) + f(x_2, x_3) + \ldots + f(x_2, x_n) +$

$f(x_3, x_1) + f(x_3, x_2) + \ldots + f(x_3, x_n) + \ldots$

$= 0$

2. For all $X, Y \subseteq V$, we have $f(X, Y) = - f(Y, X)$.

$$f(X, Y) = \sum_{x \in X} \sum_{y \in Y} f(x, y)$$
$$= \sum_{y \in Y} \sum_{x \in X} -f(y, x)$$
$$= - f(Y, X)$$

3. For all $X, Y, Z \subseteq V$ with $X \cap Y = ø$, we have the sums

$$f(X \cup Y, Z) = f(X, Z) + f(Y, Z),$$
$$f(Z, X \cup Y) = f(Z, X) + f(Z, Y).$$

$$f(X \cup Y, Z) = \sum_{x \in X \cup Y} \sum_{z \in Z} f(x, z)$$
$$= \sum_{x \in Z} \sum_{z \in Z} f(x, z) + \sum_{y \in Y} \sum_{z \in Z} f(y, z)$$
$$= f(X, Z) + f(Y, Z)$$

- Working with flows
  - $|f| = f(V, t)$

    $|f| = f(s, V)$

    $= f(V, V) - f(V - s, V)$

    $= - f(V - s, V)$

    $= f(V, V - s)$

    $= f(V, t) + f(V, V - s - t)$

    $= f(V, t)$

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   $$f(X \cup Y, Z) = f(X, Z) + f(Y, Z),$$

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- The Ford-Fulkerson method
  - *The maximum-flow problem*: given a flow network *G* with source *s* and sink *t*, we wish to find a flow *f* of maximum value. ($\sum_{u \in V, f(u,v)>0} f(u,v)$)
  - important concepts:

    residual networks

    augmenting paths

    cuts

Ford-Fulkerson-Method(*G*, *s*, *t*)

1. Initialize flow *f* to 0
2. **while** there exists an augmenting path *p* in the current residue graph
3. **do** augment flow *f* along *p*
4. **return** *f*

- Residual networks
  - Given a flow network and a flow, the ***residual network*** consists of edges that can admit more flow.
  - Let f be a flow in $G = (V, E)$ with source s and sink t. Consider a pair of vertices $u, v \in V$. The amount of *additional* flow we can push from u to v before exceeding the capacity $c(u, v)$ is the ***residual capacity*** of (u, v), given by

$$c_f(u, v) = c(u, v) - f(u, v).$$

  - Example

    If $c(u, v) = 16$ and $f(u, v) = 11$, then $c_f(u, v) = 16 - 11 = 5$.

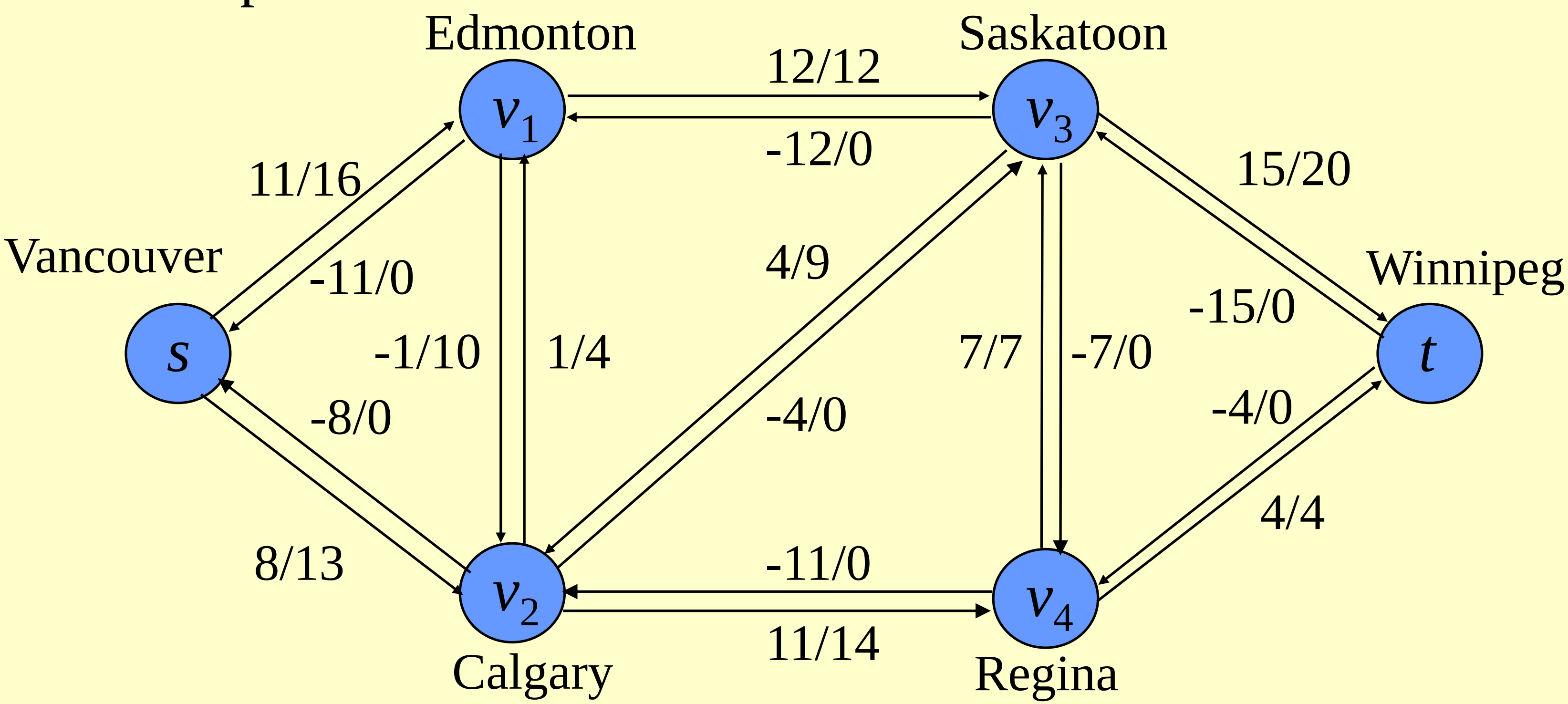
    If $c(u, v) = 17$ and $f(u, v) = -4$, then $c_f(u, v) = 17 - (-4) = 21$.

- Residual networks

  - Given a flow network $G = (V, E)$ and a flow f, the ***residual network*** of G induced by f is $G_f = (V, E_f)$, where

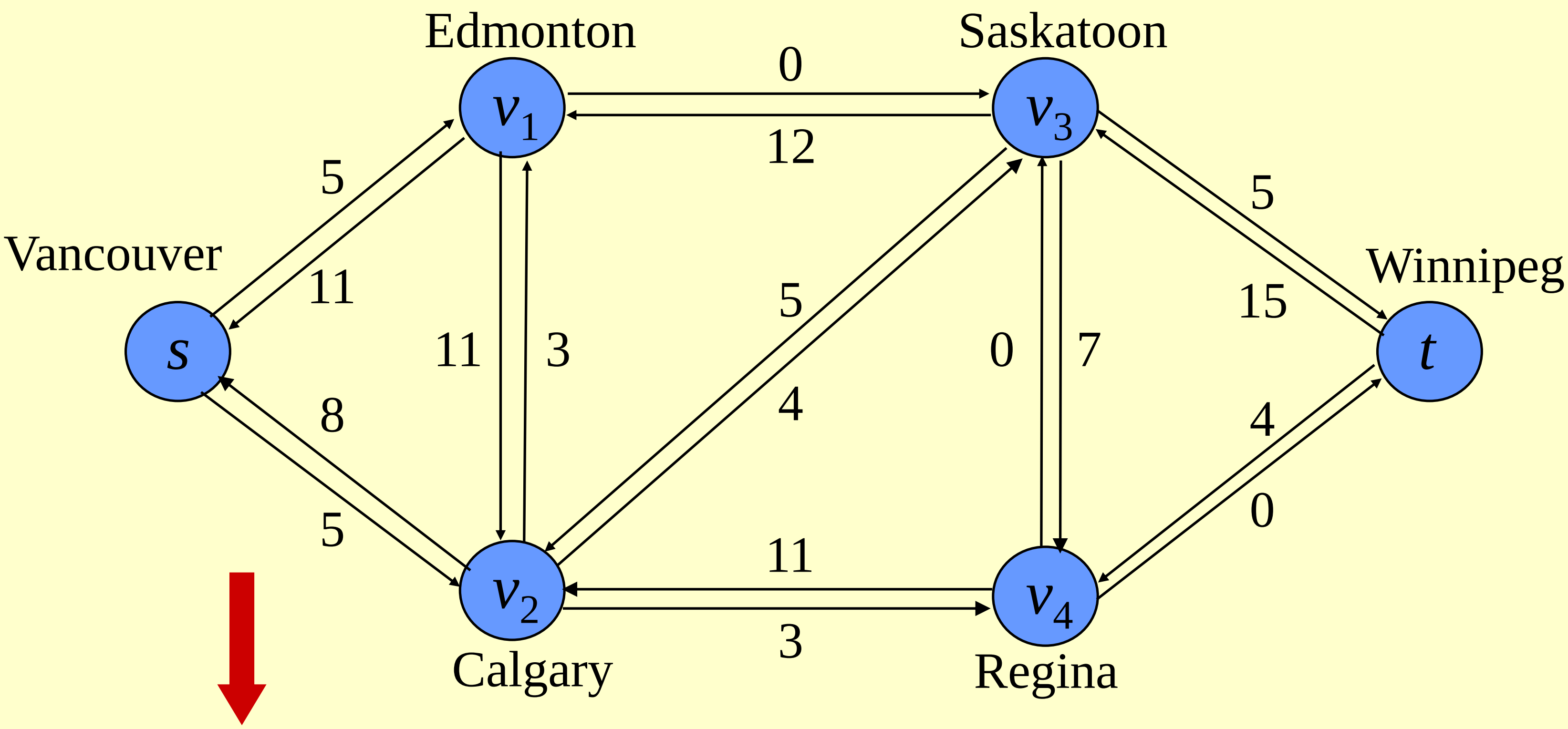
    $E_f = \{(u, v) \in V \times V: c_f(u, v) > 0\}$.

  - Example



Residual networks

residual network:



$|E_f| \leq 2|E|$

■ Residual networks

Lemma 26.2 Let $G = (V, E)$ be a network with source s and sink t, and let f be a flow in G. Let G_f be the residual network of G induced by f, and let f' be a flow in G_f. Then, the flow sum $f + f'$ (defined by $(f + f')(u, v) = f(u, v) + f'(u, v)$) is a flow in G with value $|f + f'| = |f| + |f'|$.

Proof. We must verify that the capacity constraints, skew symmetry, and flow conservation are obeyed.

Capacity constraint:

$$
\begin{aligned}
(f + f')(u, v) &= f(u, v) + f'(u, v) \\
&\leq f(u, v) + c_f(u, v) \\
&= f(u, v) + (c(u, v) - f(u, v)) \\
&= c(u, v).
\end{aligned}
$$

Skew symmetry:

$(f + f')(u, v) = f(u, v) + f'(u, v) = - f(v, u) - f'(v, u)$

$= - (f(v, u) + f'(v, u)) = - (f + f')(v, u).$

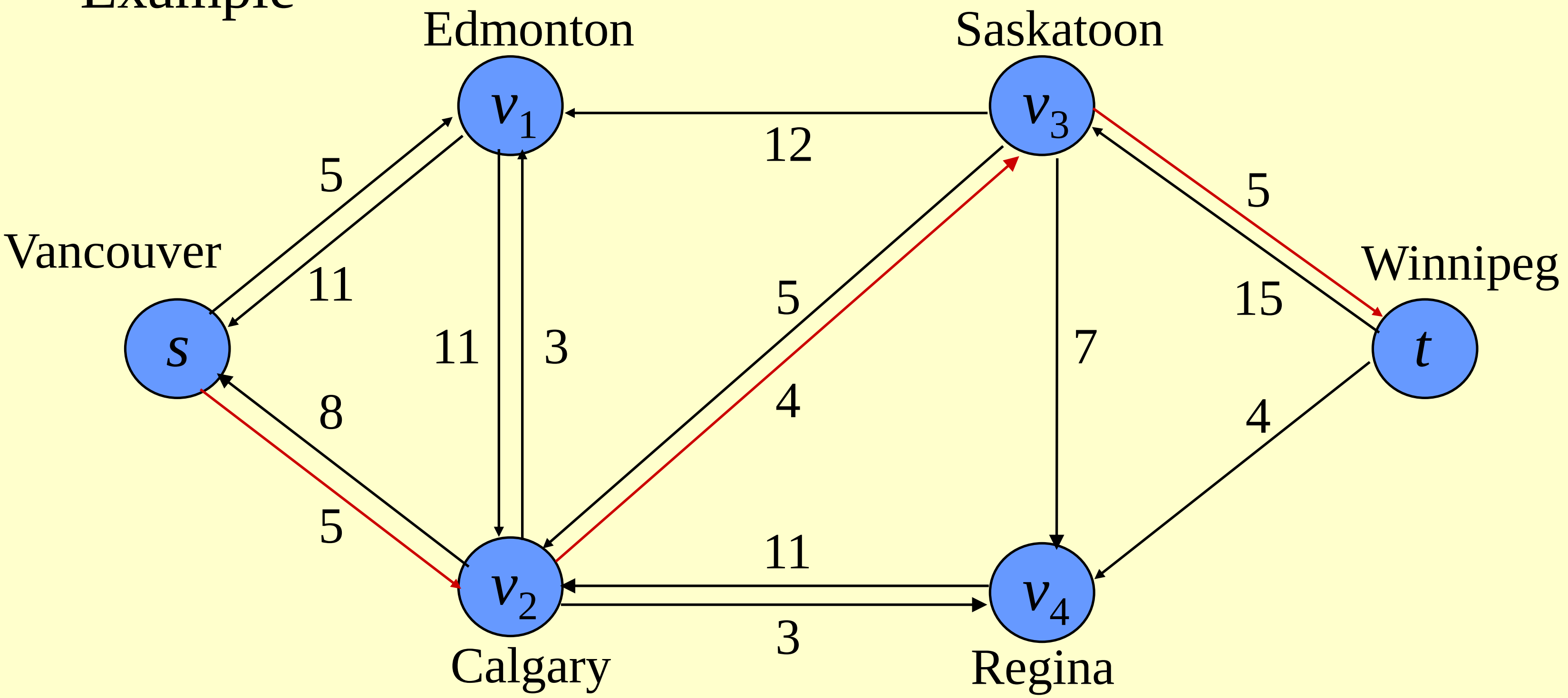
Flow conservation:

$$\begin{aligned}\sum_{v\in V}(f + f')(u,v) &= \sum_{v\in V}(f(u,v) + f'(u,v)) \\ &= \sum_{v\in V} f(u,v) + \sum_{v\in V} f'(u,v) \\ &= 0 + 0 = 0.\end{aligned}$$

Finally, we have

$$\begin{aligned}|f + f'| &= \sum_{v\in V}(f + f')(s,v) = \sum_{v\in V}(f(s,v) + f'(s,v)) \\ &= \sum_{v\in V} f(s,v) + \sum_{v\in V} f'(s,v) \\ &= |f| + |f'|\end{aligned}$$

- Augmenting paths
  - Given a flow network $G = (V, E)$ and a flow f, an *augmenting path* p is a simple path from s to t in the residual network G_f such that the residue capacity of each edge on p is > 0.
  - Example



- Augmenting paths
  - In the above residual network, path $s \rightarrow v_2 \rightarrow v_3 \rightarrow t$ is an augmenting path.
  - We can increase the flow through each edge of this path by up to 4 units without violating the capacity constraint since the smallest residual capacity on this path is $c_f(v_2, v_3) = 4$.
  - *residual capacity of an augmenting path*

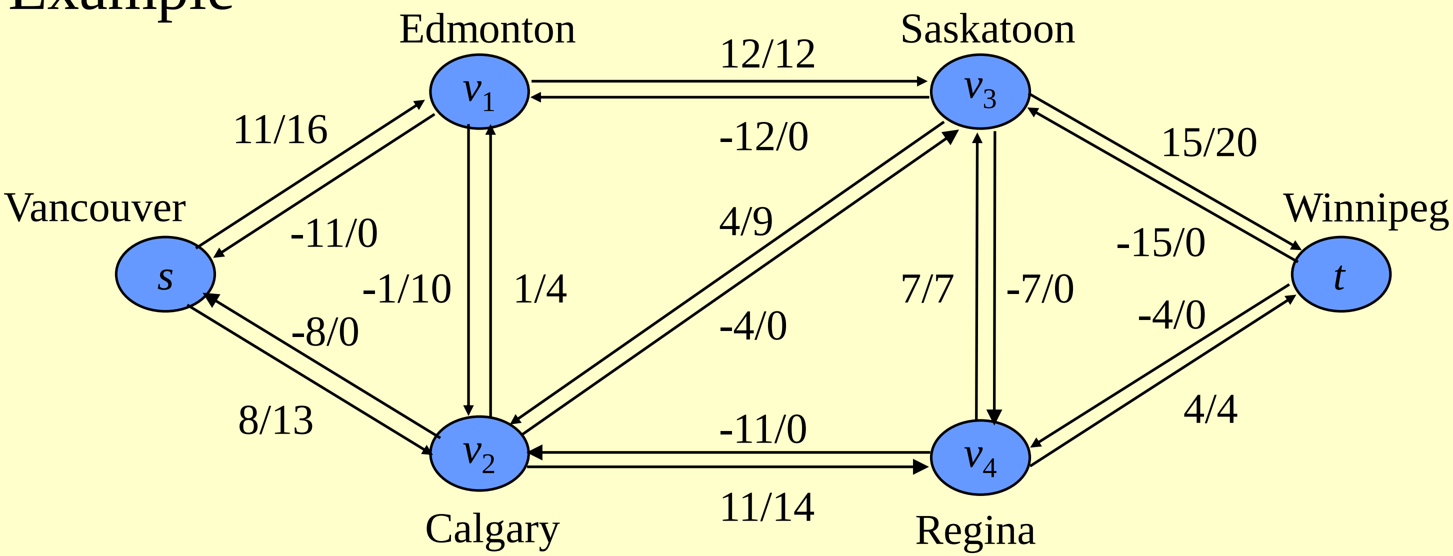
$$c_f(p) = \min\{c_f(u, v): (u, v) \text{ is on } p\}.$$

  - **Lemma 26.3** Let $G = (V, E)$ be a network, let f be a flow in G, and let p be an augmenting path in G_f. Define a function $f_p: V \times V \rightarrow \boldsymbol{R}$ by

$$f_p(u, v) = \begin{cases} c_f(p) & \text{if } (u, v) \text{ is on } p, \\ -\,c_f(p) & \text{if } (v, u) \text{ is on } p, \\ 0 & \text{otherwise.} \end{cases}$$

    Then, f_p is a flow in G_f with value $|f_p| = c_f(p)$.

- Example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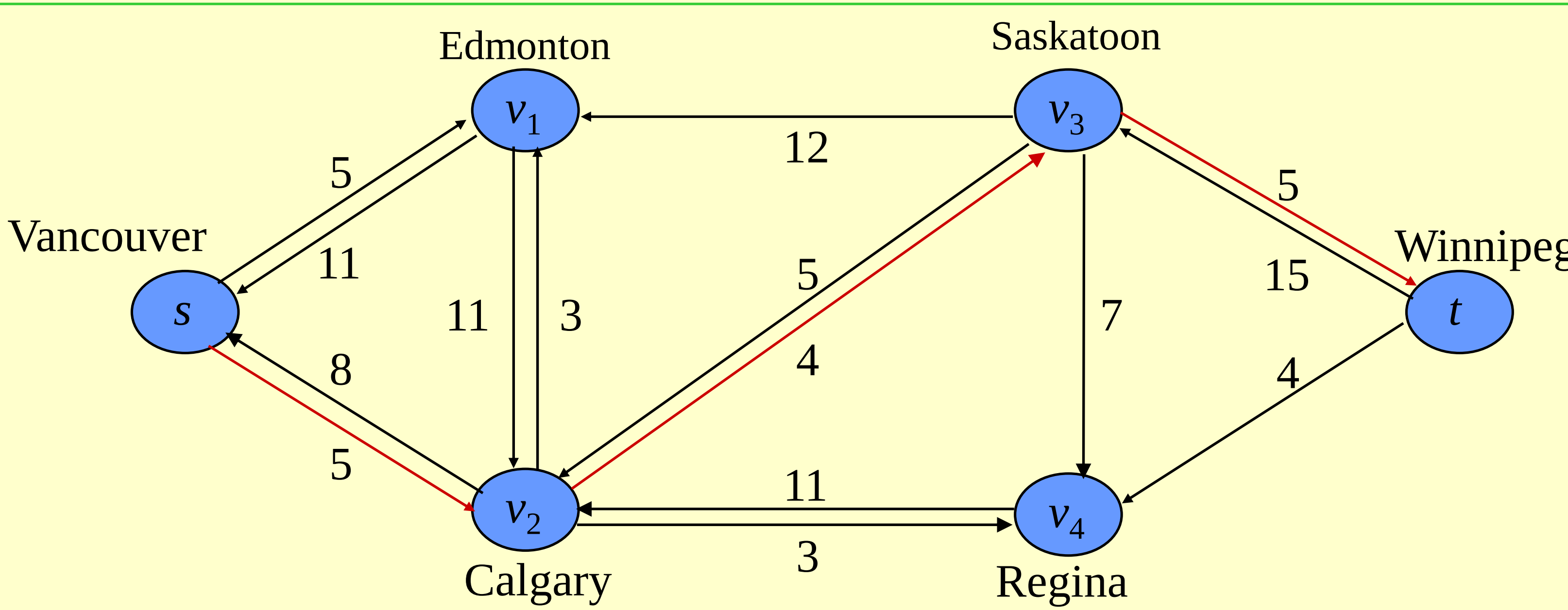
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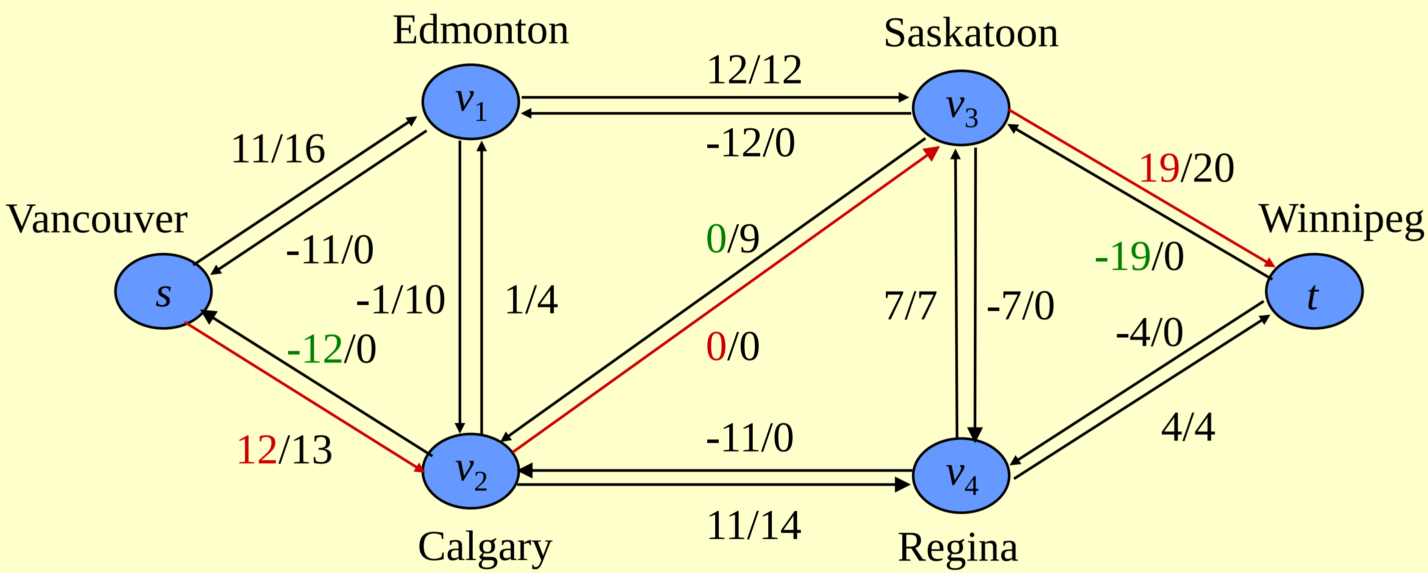
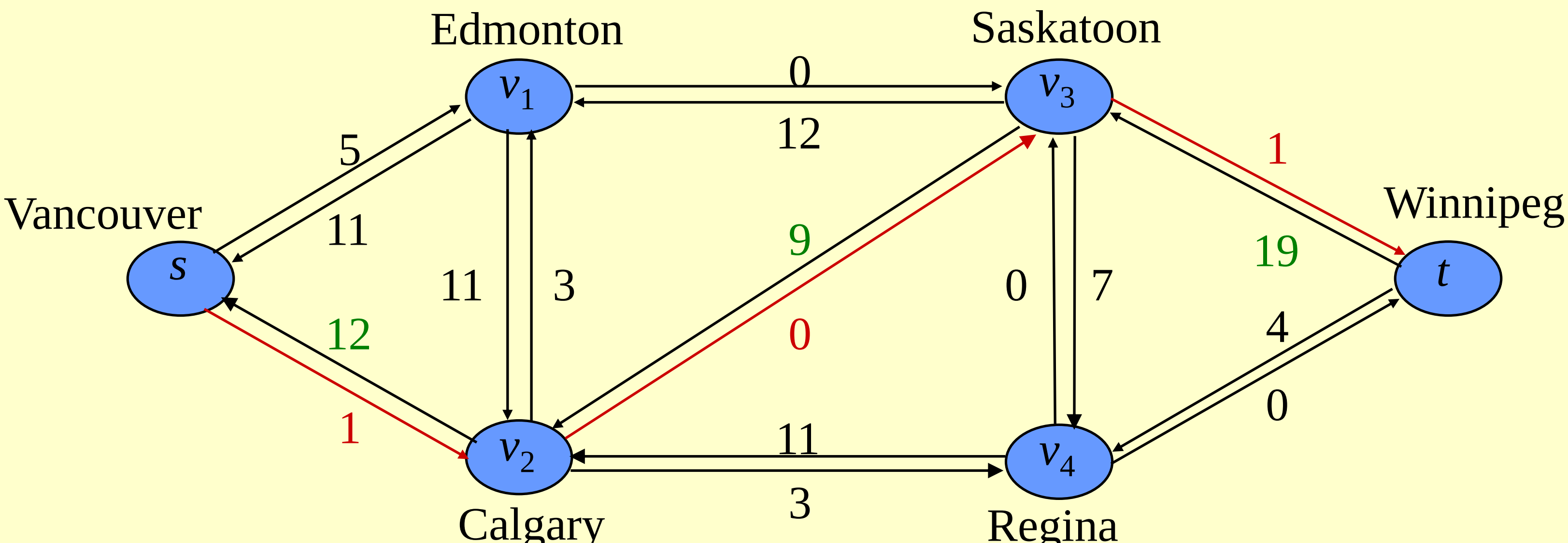
- Residual network induced by the new flow





- Augmenting paths
  - **Corollary 26.4** Let $G = (V, E)$ be a network, let f be a flow in G, and let p be an augmenting path in G_f. Let f_p be defined as in Lemma 26.3. Define a function f': $V \times V \rightarrow \boldsymbol{R}$ by

    $$f' = f + f_p.$$

    Then, f' is a flow in G with value $|f'| = |f| + |f_p| > |f|$.

    Proof. Immediately from Lemma 26.2 and 26.3.
- Ford-Fulkerson Algorithm
  - The Ford-Fulkerson method repeatedly augments the flow along augmenting paths until a maximum flow has been found.
  - A flow is maximum if and only if its residual network contains no augmenting path.

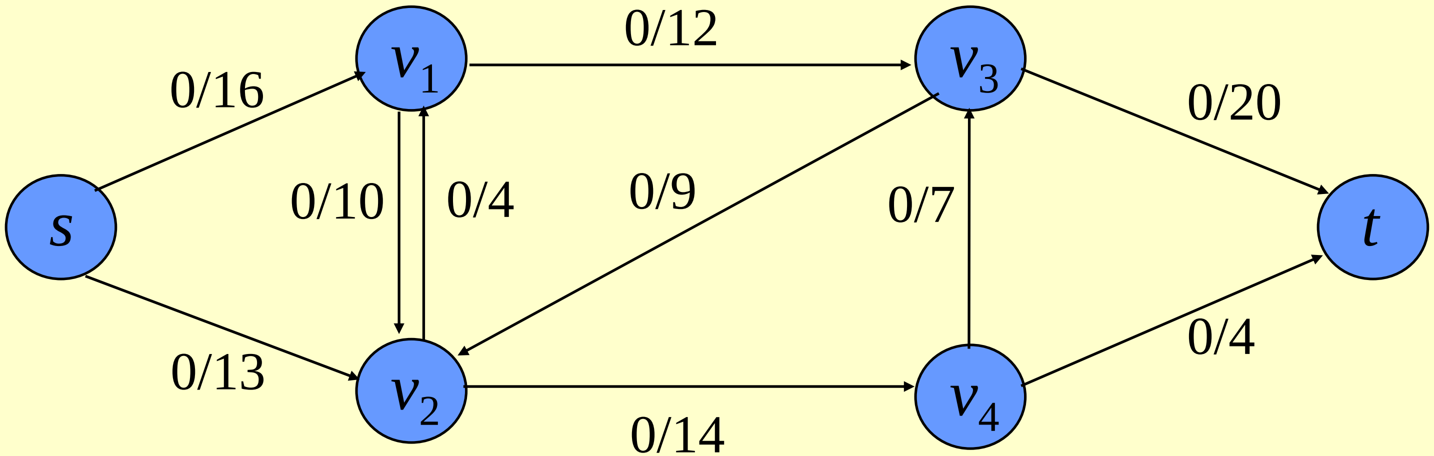
- Ford-Fulkerson algorithm

  Ford_Fulkerson(G, s, t)

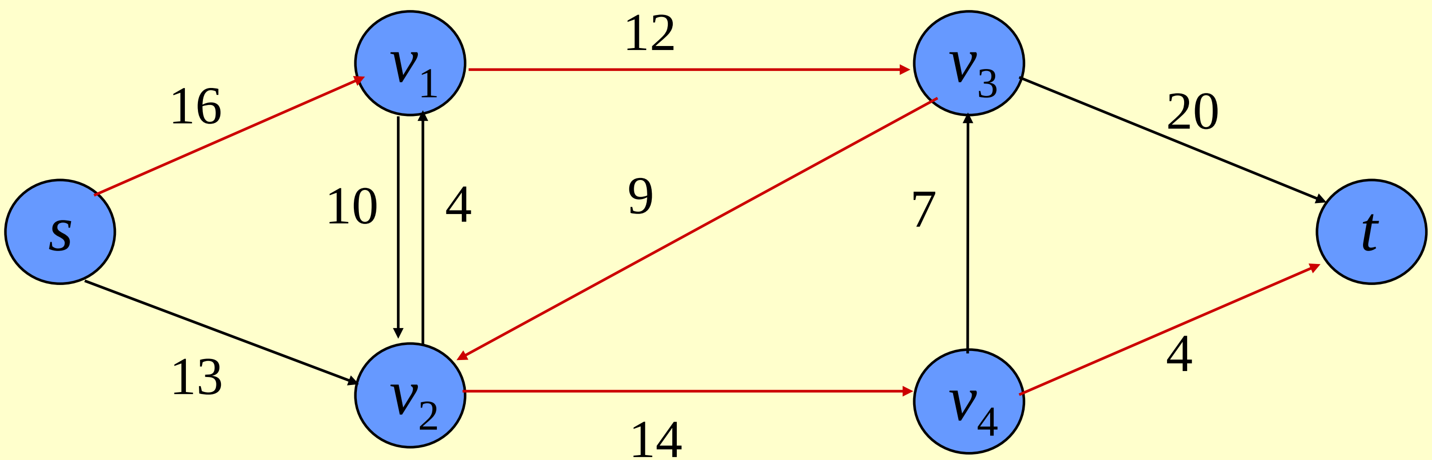
  1. **for** each edge $(u, v) \in E(G)$
  2.   **do** $f(u, v) \leftarrow 0$
  3. **while** there exists a path p from s to t in G_f
  4.   **do** $c_f(p) \leftarrow \min\{c_f(u, v) : (u, v)$ is in $p\}$
  5.   **for** each edge (u, v) on p
  6.   **do** $f(u, v) \leftarrow f(u, v) + c_f(p)$
  7. $f(v, u) \leftarrow f(v, u) - c_f(p)$

- Sample trace

Initially, the flow on edge is 0.

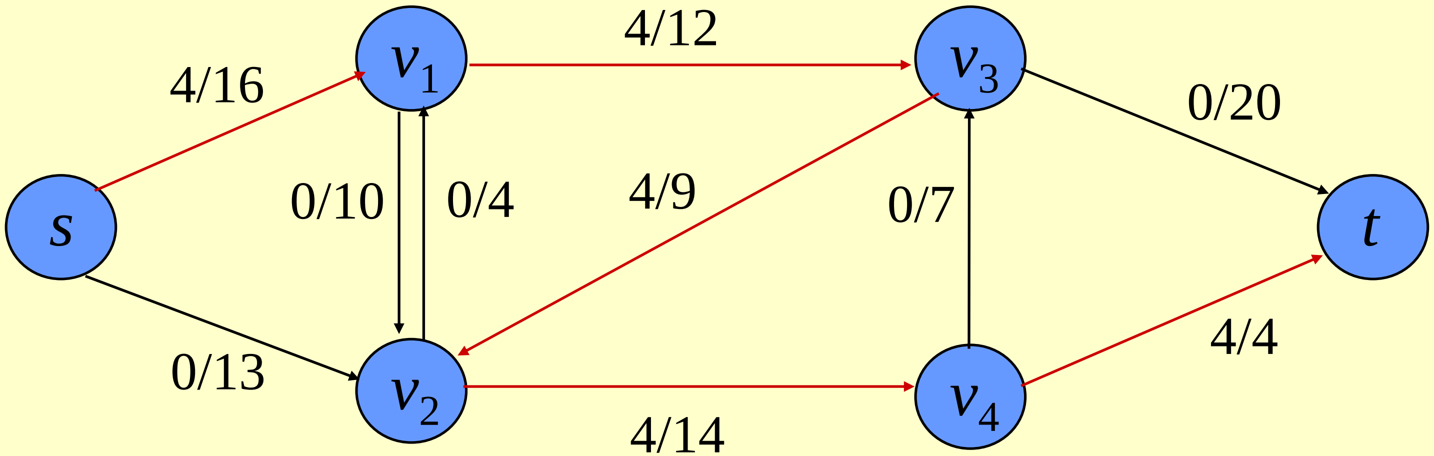


The corresponding residual network:



- Sample trace

Pushing a flow 4 on *p*1 (an augmenting path)



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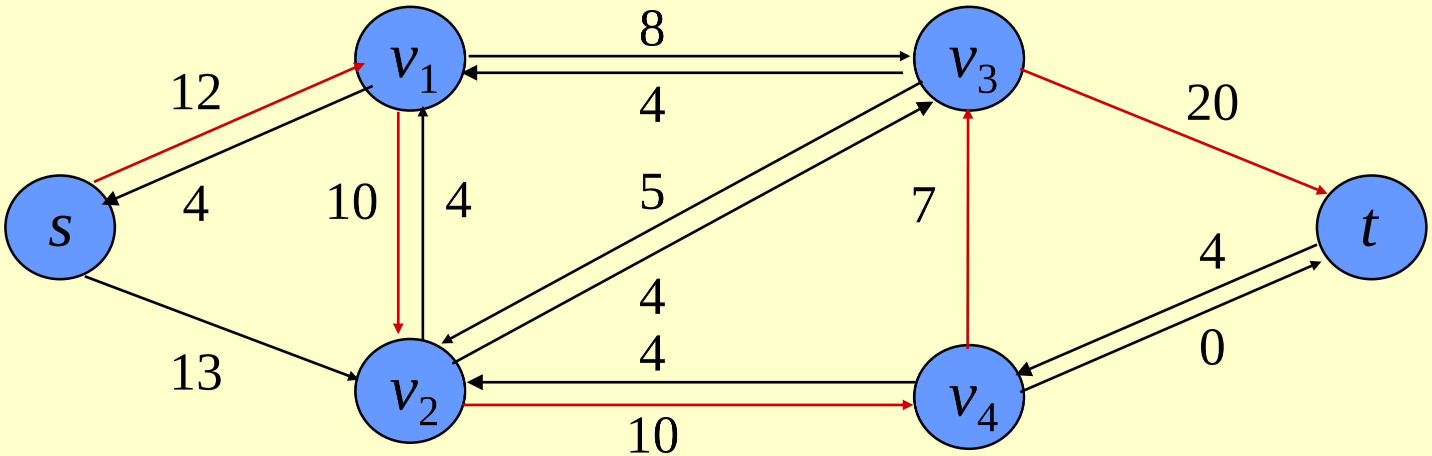
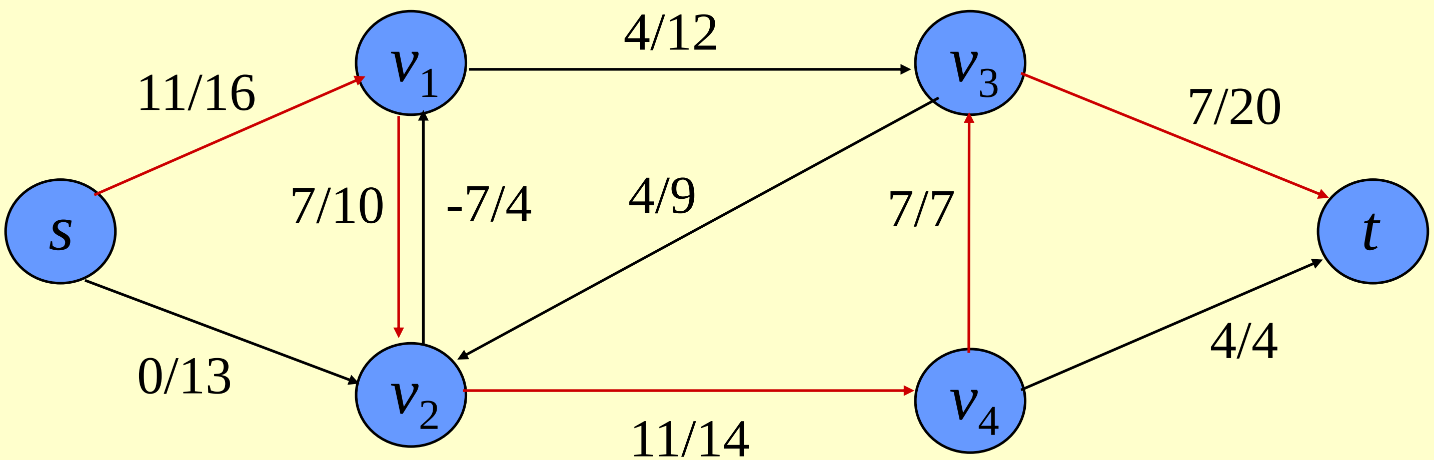
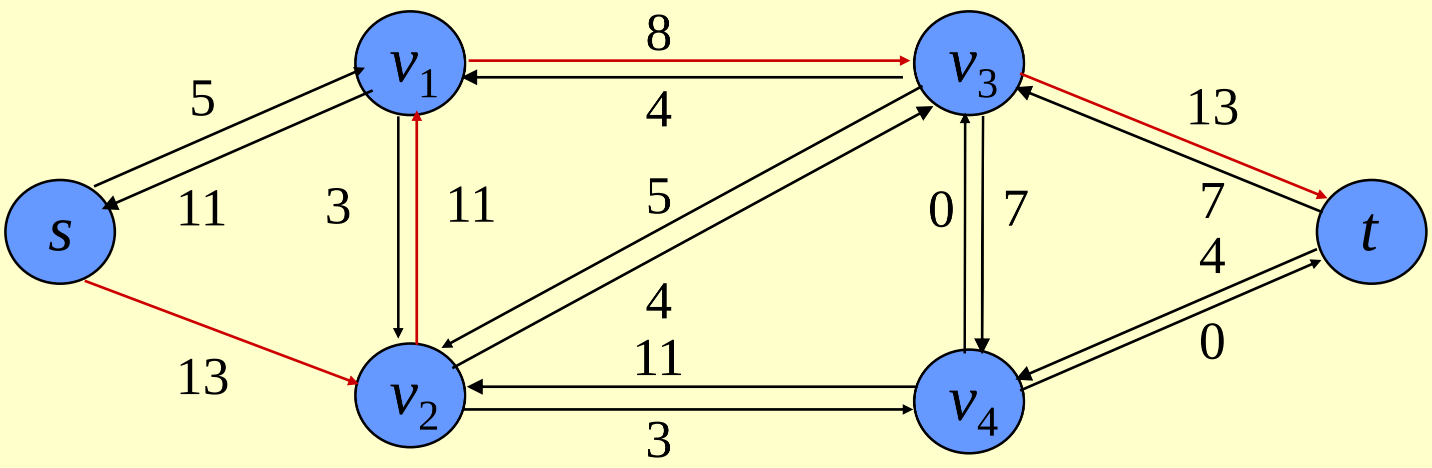


- Sample trace

Pushing a flow 7 on *p*2 (an augmenting path)

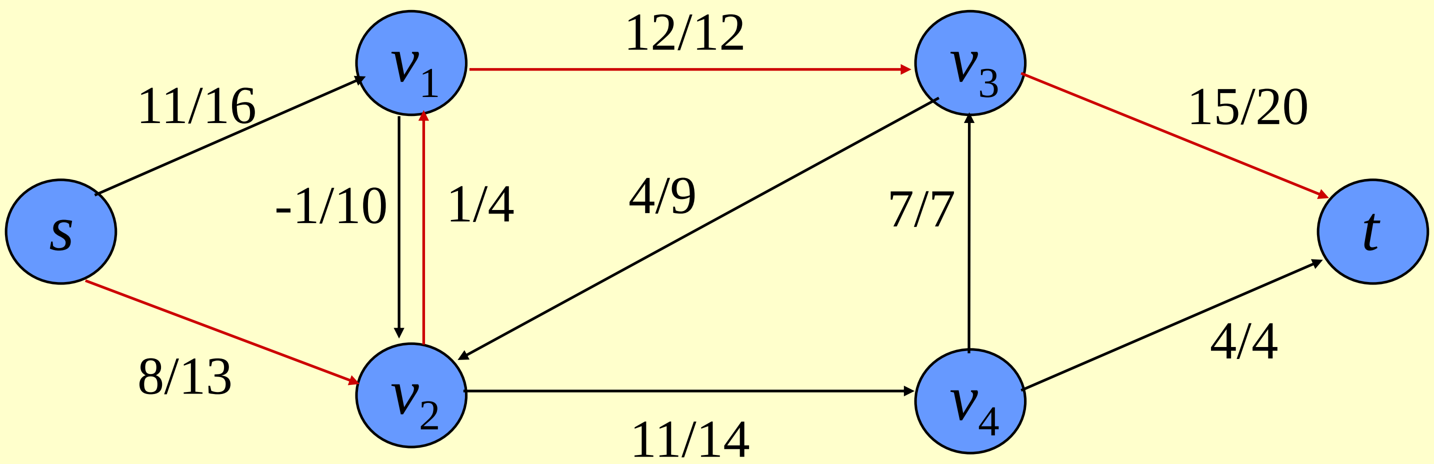


The corresponding residual network:

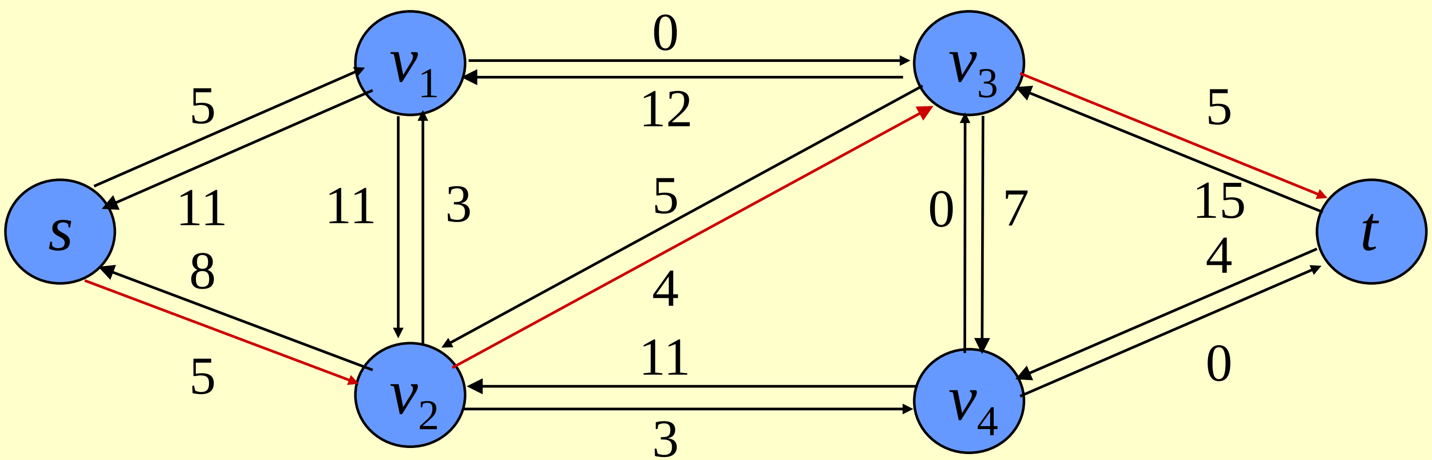


- Sample trace

Pushing a flow 8 on *p*3 (an augmenting path)

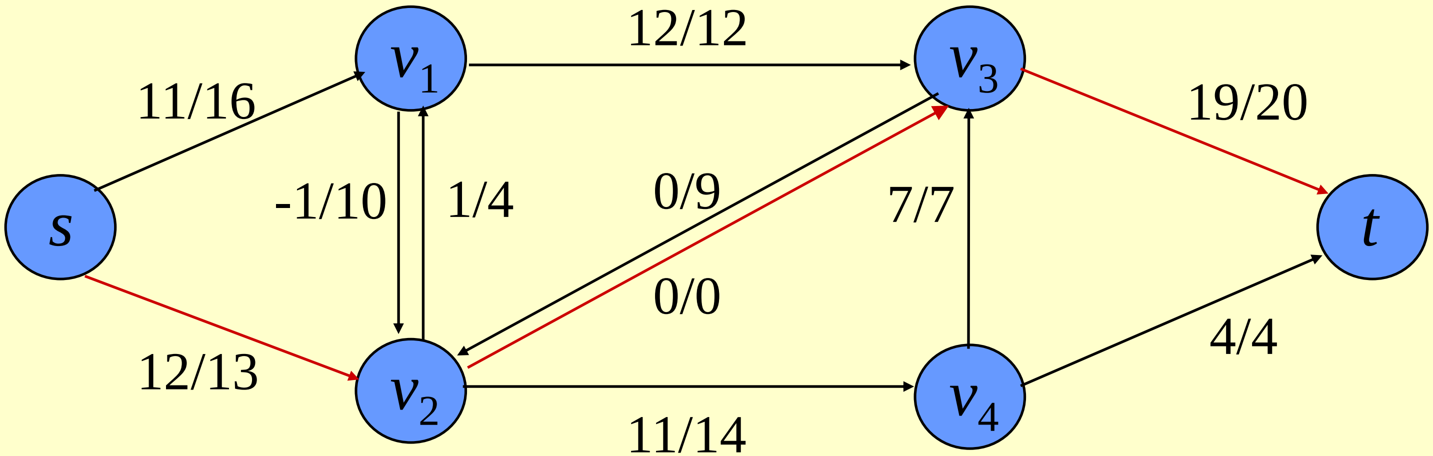


The corresponding residual network:

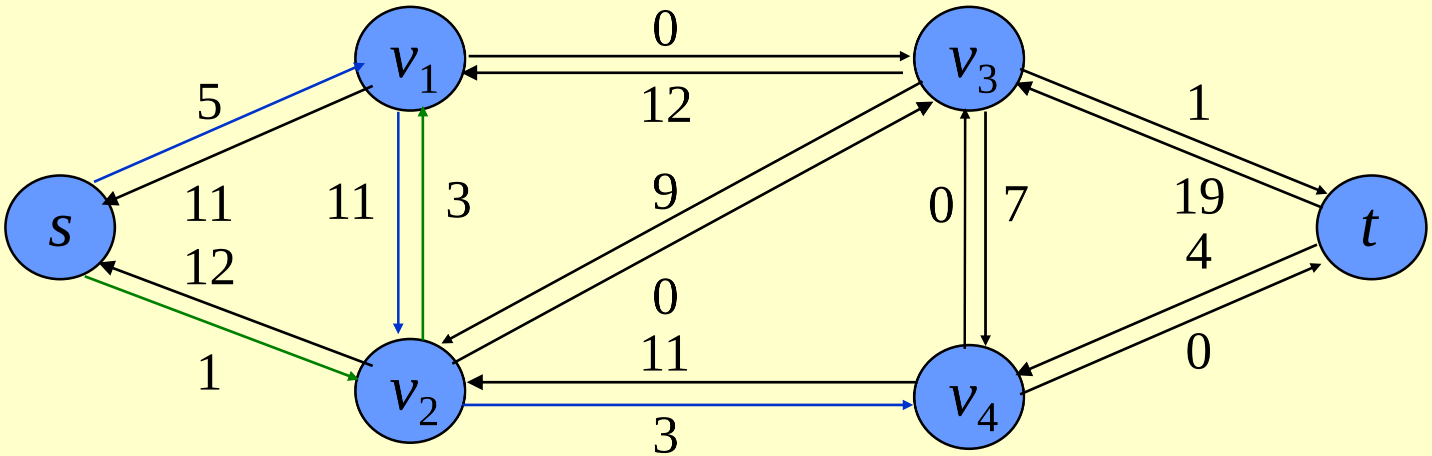


- Sample trace

Pushing a flow 4 on *p*4 (an augmenting path)



The corresponding residual network: no augmenting paths!



- Analysis of Ford-Fulkerson algorithm

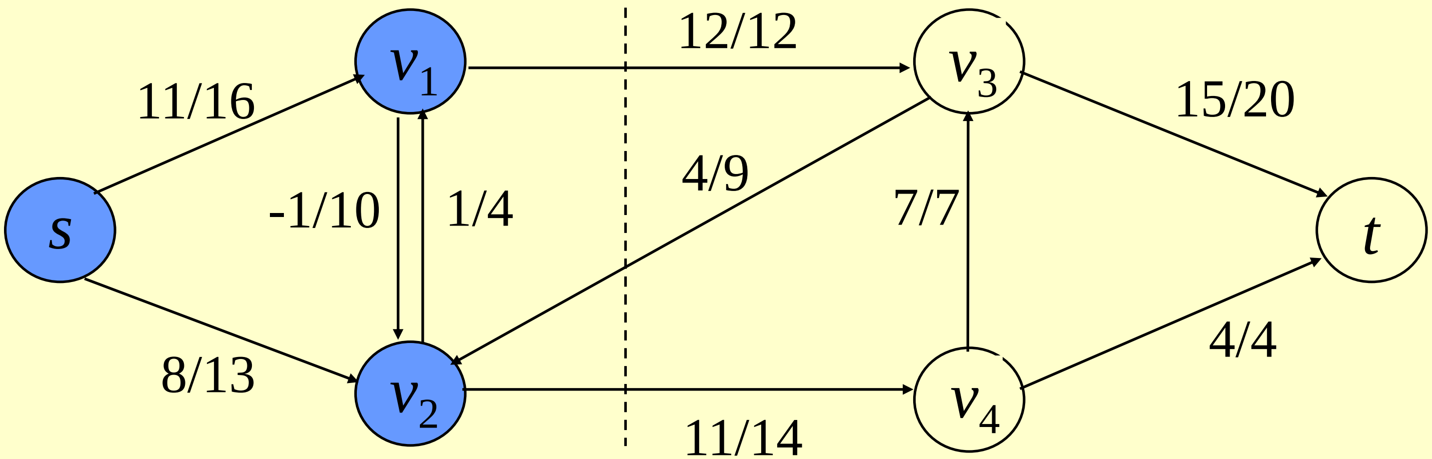
In practice, the maximum-flow problem often arises with integral capacities. If the capacities are rational numbers, an appropriate scaling transformation can be used to make them all integral. Under this assumption, a straightforward implementation of Ford-Fulkerson algorithm runs in time $O(E|f^*|)$, where f^* is the maximum flow found by the algorithm.

The analysis is as follows:

1. Lines 1-3 take time $\Theta(E)$.
2. The while-loop of lines 4-8 is executed at most $|f^*|$ times since the flow value increases by at least one unit in each iteration. Each iteration takes $O(E)$ time.

Max-flow min-cut theorem

- Cuts of flow networks
  - A cut (S, T) of flow network $G = (V, E)$ is a partition of V into S and $T = V - S$ such that $s \in S$ and $t \in T$.
  - *net flow* across the cut (S, T) is defined to be $f(S, T)$.

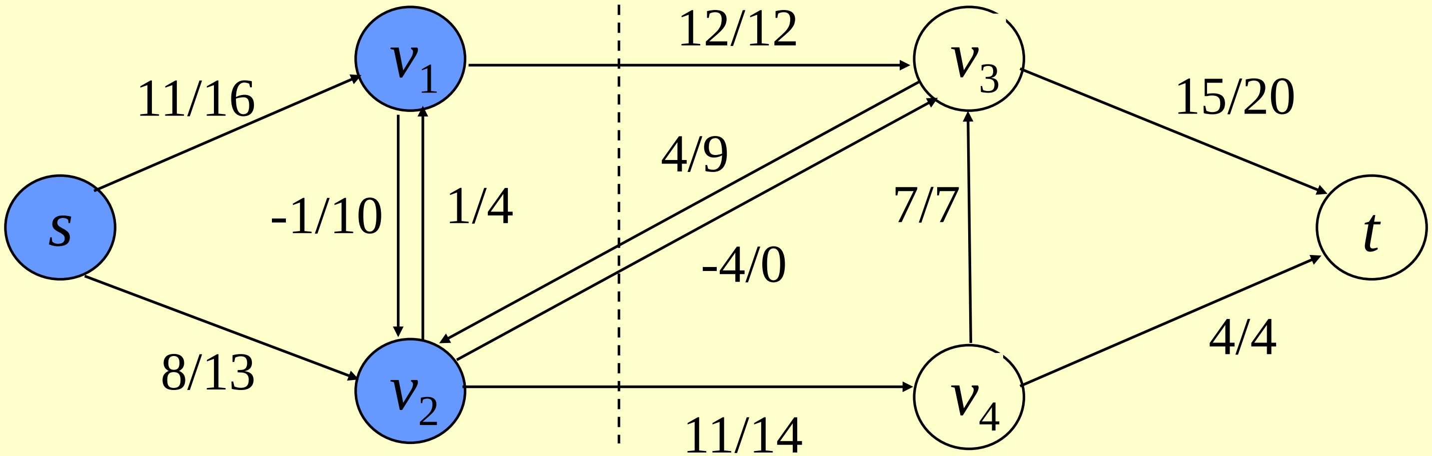


$f(\{s, v_1, v_2\}, \{v_3, v_4, t\}) = f(v_1, v_3) + f(v_2, v_3) + f(v_2, v_4)$
$= 12 + (-4) + 11 = 19.$

The net flow across a cut (S, T) consists of positive flows in both direction.

Cuts of flow networks

- The capacity of the cut (S, T) is denoted by $c(S, T)$, which is computed only from edges going from S to T.



$c(\{s, v_1, v_2\}, \{v_3, v_4, t\}) = c(v_1, v_3) + c(v_2, v_4)$
$= 12 + 14 = 26.$

- Cuts of flow networks
  - The following lemma shows that the net flow across any cut is the same, and it equals the value of the flow.

    Lemma 26.5 Let f be a flow in a flow network G with source s and sink t, and let (S, T) be a cut of G. Then, the net flow across (S, T) is $f(S, T) = |f|$.

    Proof. Note that $f(S - s, V) = 0$ by flow conservation. So we have

$$
\begin{aligned}
f(S, T) = f(S, V - S) &= f(S, V) - f(S, S) \\
&= f(S, V) \\
&= f(s, V) + f(S - s, V) \\
&= f(s, V) \\
&= |f|.
\end{aligned}
$$

- Cuts of flow networks
  - **Corollary 26.6** The value of any flow in a flow network G is bounded from above by the capacity of any cut of G.

    Proof.

$$
\begin{aligned}
|f| &= f(S, T) \\
&= \sum_{u \in S} \sum_{v \in T} f(u,v) \\
&\leq \sum_{u \in S} \sum_{v \in T} c(u,v) \\
&= c(S, T).
\end{aligned}
$$

- Max-flow min-cut theorem

Theorem 26.7 If f is a flow network $G = (V, E)$ with source s and sink t, then the following conditions are equivalent:

1. f is a maximum flow in G.

2. The residual network G_f contains no augmenting paths.

3. $|f| = c(S, T)$ for some cut (S, T) of G.

Proof. (1) $\Rightarrow$ (2): Suppose for the sake of contradiction that f is a maximum flow in G but that G_f has an augmenting path p. Then, by Corollary 26.4, the flow sum $f + f_p$, where f_p is given by Lemma 26.3, is a flow in G with value strictly greater than $|f|$, contradicting the assumption that f is a maximum flow.

- Max-flow min-cut theorem

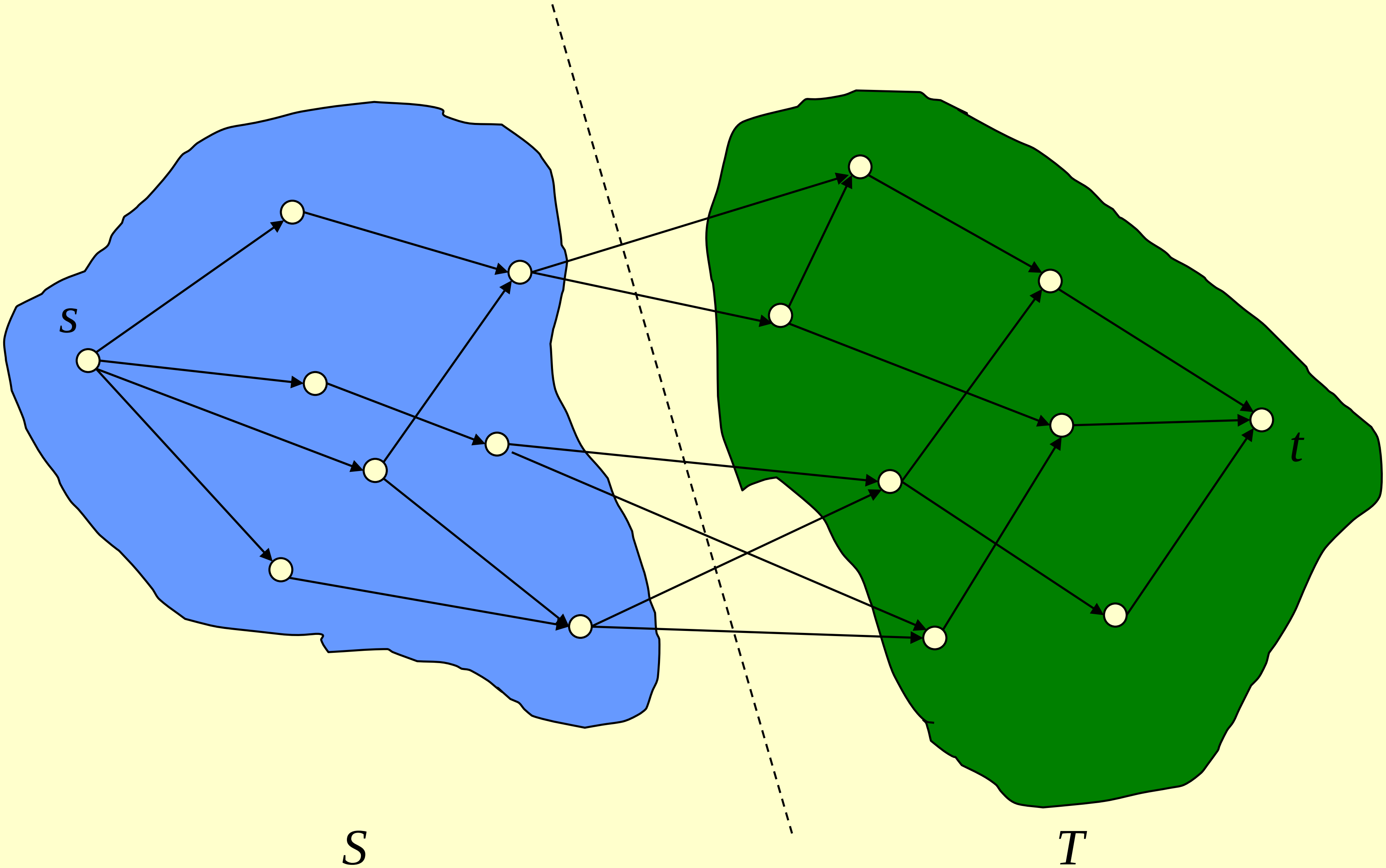
Theorem 26.7 If f is a flow network $G = (V, E)$ with source s and sink t, then the following conditions are equivalent:

1. f is a maximum flow in G.

2. The residual network G_f contains no augmenting paths.

3. $|f| = c(S, T)$ for some cut (S, T) of G.

Proof. (2) $\Rightarrow$ (3): Suppose that G_f has no augmenting path. Define $S = \{v \in V$: there exists a path from s to v in $G_f\}$ and $T = V - S$. The partition (S, T) is a cut: we have $s \in S$ trivially and $t \notin S$ because there is no path from s to t in G_f. For each pair of vertices u and v such that $u \in S$ and $v \in T$, we have $f(u, v) = c(u, v)$, since otherwise $(u, v) \in E_f$, which would place v in set S. By Lemma 26.5, therefore, $|f| = f(S, T) = c(S, T)$.



s
t
S
T

- Max-flow min-cut theorem

Theorem 26.7 If f is a flow network $G = (V, E)$ with source s and sink t, then the following conditions are equivalent:

1. f is a maximum flow in G.
2. The residual network G_f contains no augmenting paths.
3. $|f| = c(S, T)$ for some cut (S, T) of G.

Proof. (3) $\Rightarrow$ (1): By Corollary 26.6, $|f| \leq c(S, T)$ for all cuts (S, T). The condition $|f| = c(S, T)$ thus implies that f is a maximum flow.