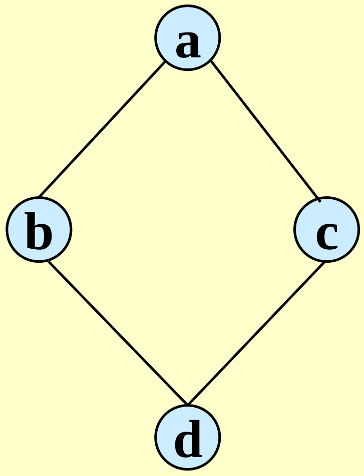
Elementary Graph Algorithms

- Graph representation
- Graph traversal
  - Breadth-first search
  - Depth-first search
- Parenthesis theorem

Graphs

- *Graph G = (V, E)*
  - » V = set of vertices
  - » E = set of edges $\subseteq (V \times V)$



$V = \{a, b, c, d\}$

$E = \{(a, b), (a, c), (b, d), (c, d)\}$

Graphs

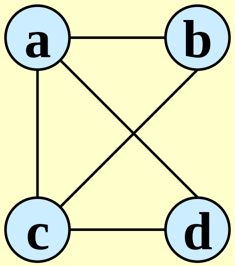
- Types of graphs
  - » Undirected: edge $(u, v) = (v, u)$; for all v, $(v, v) \notin E$ (No self loops.)
  - » Directed: (u, v) is edge from u to v, denoted as $u \to v$. Self loops are allowed.
  - » Weighted: each edge has an associated weight, given by a weight function $w : E \to R$. (R – set of all possible real numbers)
  - » Dense: $|E| \approx |V|^2$.
  - » Sparse: $|E| << |V|^2$.
- $|E| = O(|V|^2)$

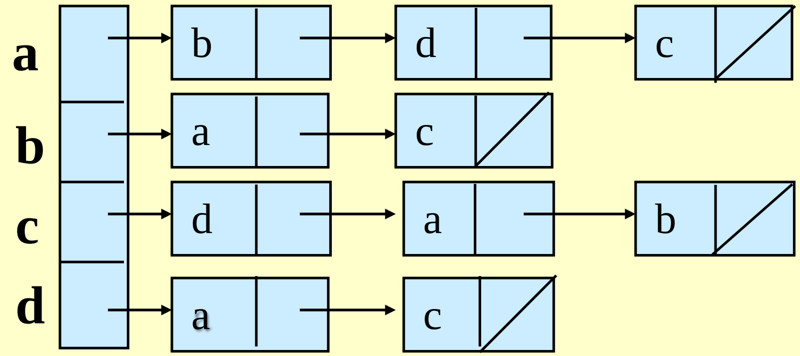
Graphs

- If $(u, v) \in E$, then vertex v is adjacent to vertex u.
- Adjacency relationship is:
  - » Symmetric if G is undirected.
  - » Not necessarily so if G is directed.
- If an undirected graph G is connected:
  - » There is a path between every pair of vertices.
  - » $|E| \geq |V| - 1$.
  - » Furthermore, if $|E| = |V| - 1$, then G is a *tree*.
  - If a directed graph G is connected:
  - » Its undirected version is connected.
  - Other definitions in Appendix B (B.4 and B.5) as needed.

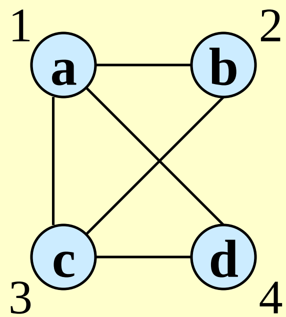
Representation of Graphs

- Two standard ways.
  - » Adjacency Lists.





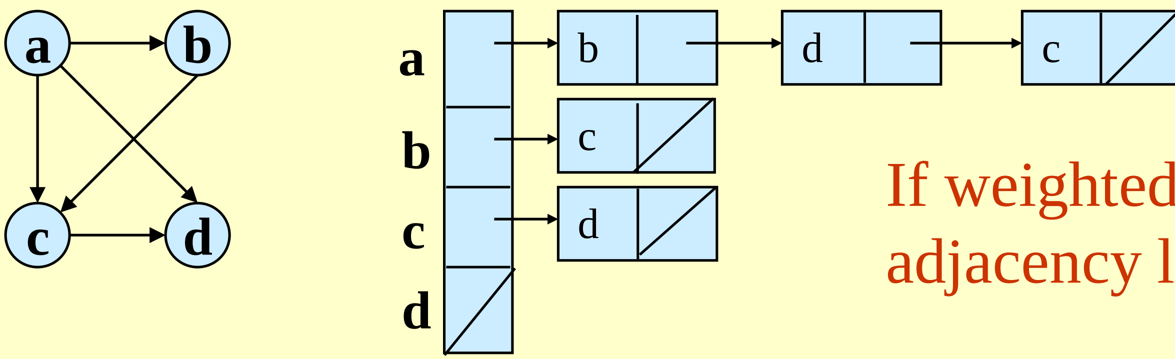
  - » Adjacency Matrix.



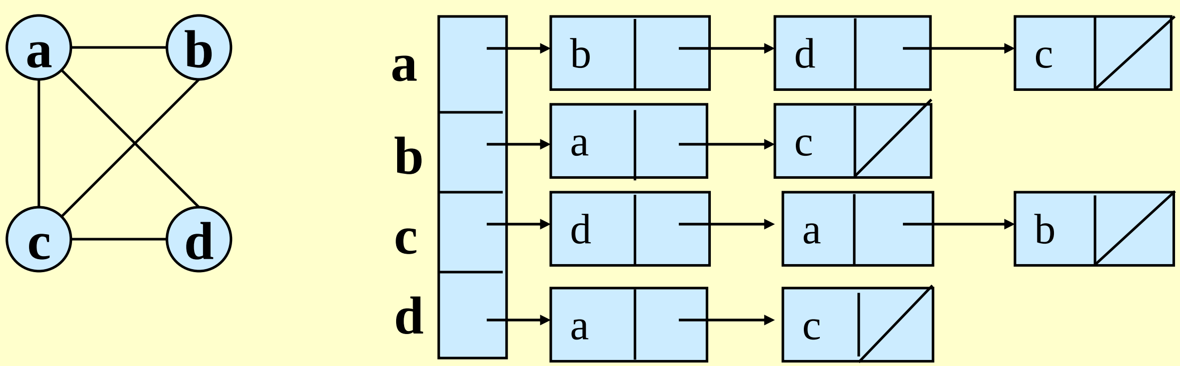
	1	2	3	4
1	0	1	1	1
2	1	0	1	0
3	1	1	0	1
4	1	0	1	0

Adjacency Lists

- Consists of an array *Adj* of $|V|$ lists.
- One list per vertex.
- For $u \in V$, $Adj[u]$ consists of all vertices adjacent to u.



If weighted, store weights also in adjacency lists.



Storage Requirement

- For directed graphs:
  - » Sum of lengths of all adj. lists is

    $$\sum_{v \in V} \text{out-degree}(v) = \sum_{v \in V} \text{in-degree}(v) = |E|$$

    No. of edges leaving *v*
  - » Total storage: $\Theta(|V| + |E|)$
- For undirected graphs:
  - » Sum of lengths of all adj. lists is

    $$\sum_{v \in V} \text{degree}(v) = 2|E|$$

    No. of edges incident on *v*. Edge (*u*,*v*) is incident on vertices *u* and *v*.
  - » Total storage: $\Theta(|V| + |E|)$

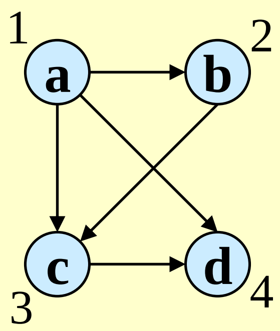
Pros and Cons: adj list

- Pros
  - » Space-efficient, when a graph is sparse.
  - » Can be modified to support many graph variants.
- Cons
  - » Determining if an edge $(u, v) \in G$ is not efficient.
    - Have to search in u's adjacency list. $\Theta(\text{degree}(u))$ time.
    - $\Theta(|V|)$ in the worst case.

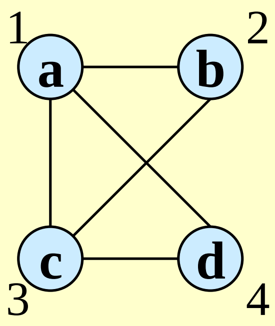
Adjacency Matrix

- $|V| \times |V|$ matrix A.
- Number vertices from 1 to $|V|$ in some arbitrary manner.
- A is then given by:

$$A[i,j] = a_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$



	1	2	3	4
1	0	1	1	1
2	0	0	1	0
3	0	0	0	1
4	0	0	0	0



	1	2	3	4
1	0	1	1	1
2	1	0	1	0
3	1	1	0	1
4	1	0	1	0

$A = A^T$ for undirected graphs.

Space and Time

- **Space:** $\Theta(|V|^2)$.
  - » Not memory efficient for large graphs.
- **Time:** to list all vertices adjacent to u: $\Theta(|V|)$.
- **Time:** to determine if $(u, v) \in E$: $\Theta(1)$.
- Can store weights instead of bits for weighted graph.

Sparse Matrix

- Sparse matrices are typically stored in a format, or a representation, which avoids storing zero elements.
- CSR (Compressed Sparse Row) - store only non-zero values in a a one-dimentional data storage: data[ ].
- Two auxiliary data structures col_index[ ] and row_ptr[ ] to preserve the stucture of the original sparse matrix in the compressed representation.
- col_index[ ] gives the column index of every nonzero value in the original sparse matrix.
- Row_ptr[ ] indicates the starting nonzero location of every row in the compressed format.

	0	1	2	3
0	3	0	1	0
1	0	0	0	0
2	0	2	4	1
3	1	0	0	1

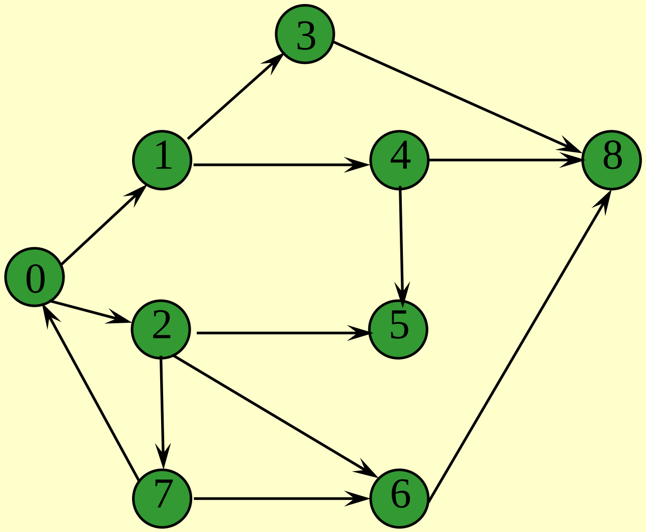
Sparse Matrix

	0	1	2	3
0	3	0	1	0
1	0	0	0	0
2	0	2	4	1
3	1	0	0	1

	row0		row2			row3	
Nonzero values data[ ]	{3	1	2	4	1	1	1}
Column indeces col_index[ ]	{0	2	1	2	3	0	3}
Row pointers row_ptr[ ]	{0	2	2	5	7}		

- In data[ ], value 3 and 1 came from column 0 and 2 in the original sparse matrix. The col_index[0] and col_index[1] elements are assigned to store the column indices for these two values. For another example, values 2, 4, and 1 came from column 1, 2, and 3 of row 2 in the original sparse matrix. Therefore, col_index[2], col_index[3], and col_index[4] store indices 1, 2, and 3.
- In row_ptr[ ], the values are the indices for the beginning locations of each row. For example, row_ptr[0] = 0 indicates the row 0 starts at location 0 of data[ ]. row_ptr[2] = 2 indicates the row 2 starts at location 2 of data[ ]. But we notice that row_ptr{1] is set to be 2, equal to row_ptr[2], showing all elements in row 1 in the original matrx are 0. Finally, row_ptr[4] stores the starting location of a non-existing 'row 4'. (This choice is the convenience, as some algorithms need to use the starting location of the next row to delineate the end of the current row.)

Sparse Graph



	0	1	2	3	4	5	6	7	8
0	0	1	1	0	0	0	0	0	0
1	0	0	0	1	1	0	0	0	0
2	0	0	0	0	0	1	1	1	0
3	0	0	0	0	1	0	0	0	1
4	0	0	0	0	0	1	0	0	1
5	0	0	0	0	0	0	1	0	0
6	0	0	0	0	0	0	0	0	1
7	0	0	0	0	0	0	1	0	0
8	0	0	0	0	0	0	0	0	0

Nonzero values data[ ] 1 1 1 1 1 1 1 1 1 1 1 1 1 1

destination[ ] 1 2 3 4 5 6 7 4 8 5 8 6 8 0 8

edges[ ] 0 2 4 7 9 11 12 13 15 16

Graph-searching Algorithms

- Searching a graph:
  - » Systematically follow the edges of a graph to visit the vertices of the graph.
- Used to discover the structure of a graph.
- Standard graph-searching algorithms.
  - » Breadth-first Search (BFS).
  - » Depth-first Search (DFS).

Breadth-first Search

- **Input:** Graph $G = (V, E)$, either directed or undirected, and ***source vertex*** $s \in V$.
- **Output:**
  - » $d[v]$ = distance (smallest # of edges, or shortest path) from s to v, for all $v \in V$. $d[v] = \infty$ if v is not reachable from s.
  - » $\pi[v] = u$ such that (u, v) is last edge on shortest path $s \leadsto v$.
    - u is v's predecessor.
  - » Builds breadth-first tree with root s that contains all reachable vertices.

Definitions:

Path between vertices u and v: Sequence of vertices $(v_1, v_2, \ldots, v_k)$ such that $u = v_1$ and $v = v_k$, and $(v_i, v_{i+1}) \in E$, for all $1 \leq i \leq k-1$.

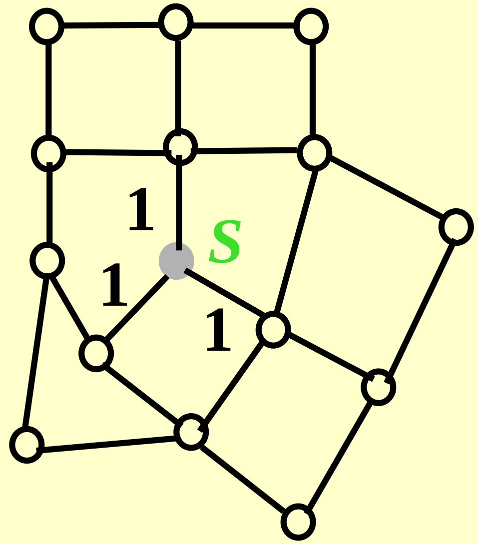
Length of the path: Number of edges in the path.

Path is simple if no vertex is repeated.

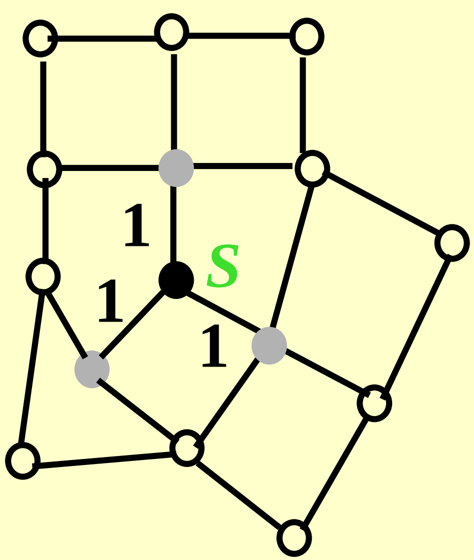
Breadth-first Search

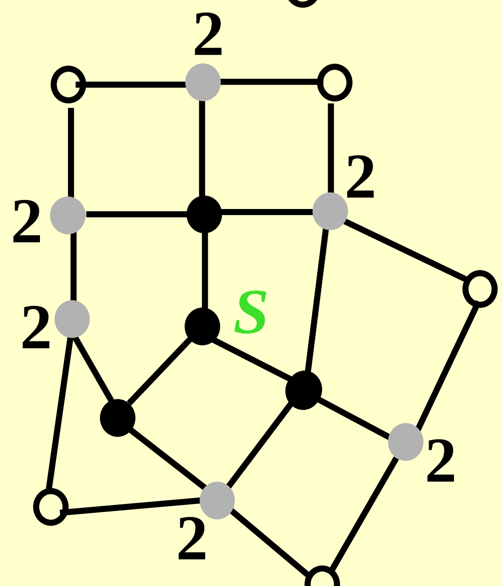
- Expands the frontier between discovered and undiscovered vertices uniformly across the breadth of the frontier.
  - » A vertex is "discovered" the first time it is encountered during the search.
  - » A vertex is "finished" if all vertices adjacent to it have been discovered.
- Colors the vertices to keep track of progress.
  - » White – Undiscovered.
  - » Gray – Discovered but not finished.
  - » Black – Finished.

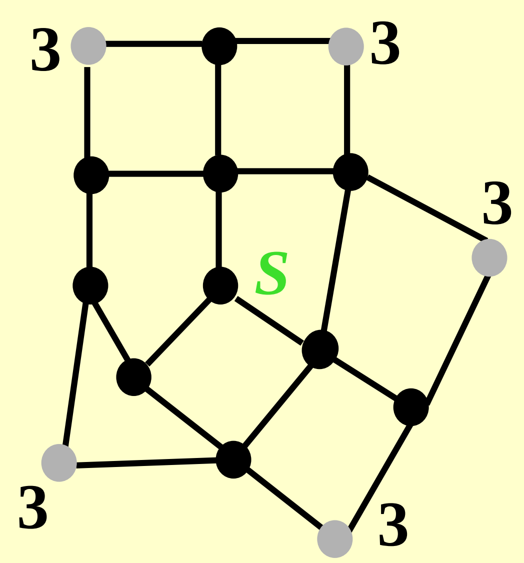
BFS for Shortest Paths



- Finished
- Discovered
- Undiscovered







BFS(G,s)

```
for each vertex u in V[G] – {s}
        do color[u] ← white
            d[u] ← ∝
            π[u] ← nil
color[s] ← gray
d[s] ← 0
 π[s] ← nil
Q ← Φ
enqueue(Q, s)
while Q ≠ Φ
        do u ← dequeue(Q)
        for each v in Adj[u] do
        if color[v] = white
        then color[v] ← gray
        d[v] ← d[u] + 1
        π[v] ← u
        enqueue(Q, v)
        color[u] ← black
```

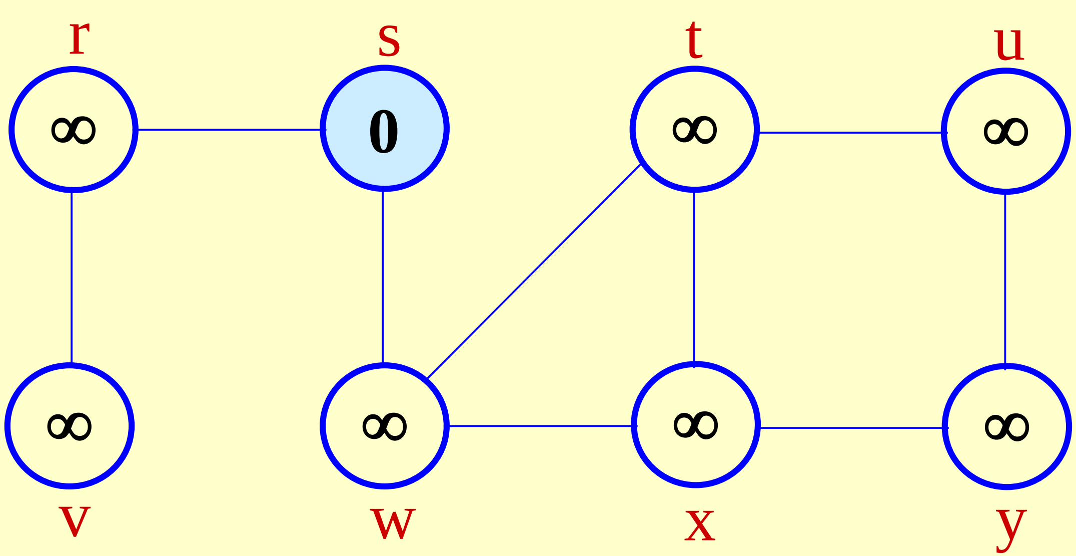
initialization

access source *s*

white: undiscovered
gray: discovered
black: finished

Q: a queue of discovered vertices
color[*v*]: color of v
d[*v*]: distance from s to v
π[*u*]: predecessor of v

Example (BFS)



Q: s

0

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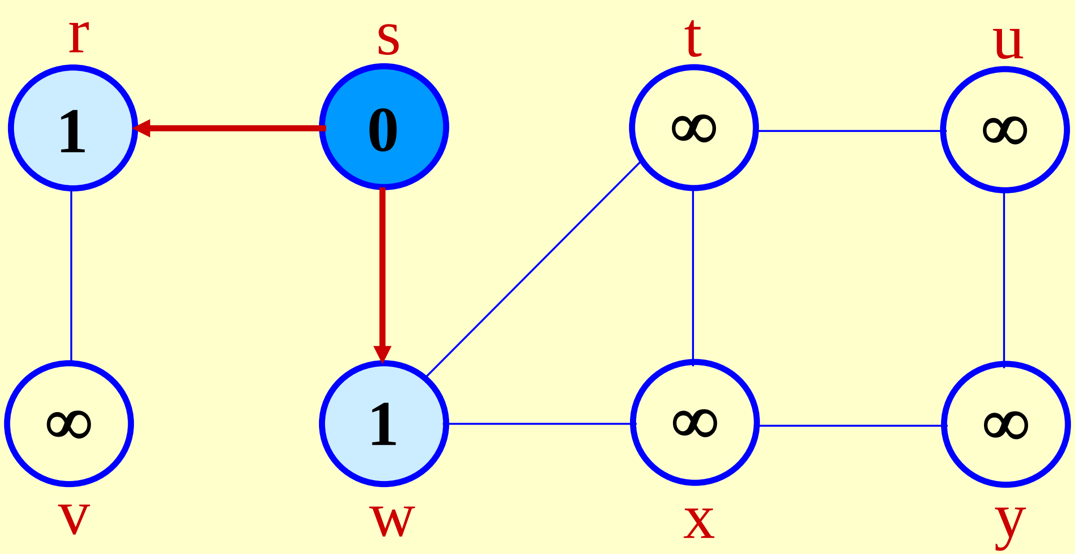
        enqueue(Q, v)
        color[u] ← black
```

Example (BFS)



Q: w r
1 1

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        π[v] ← u
        enqueue(Q, v)
        color[u] ← black
```
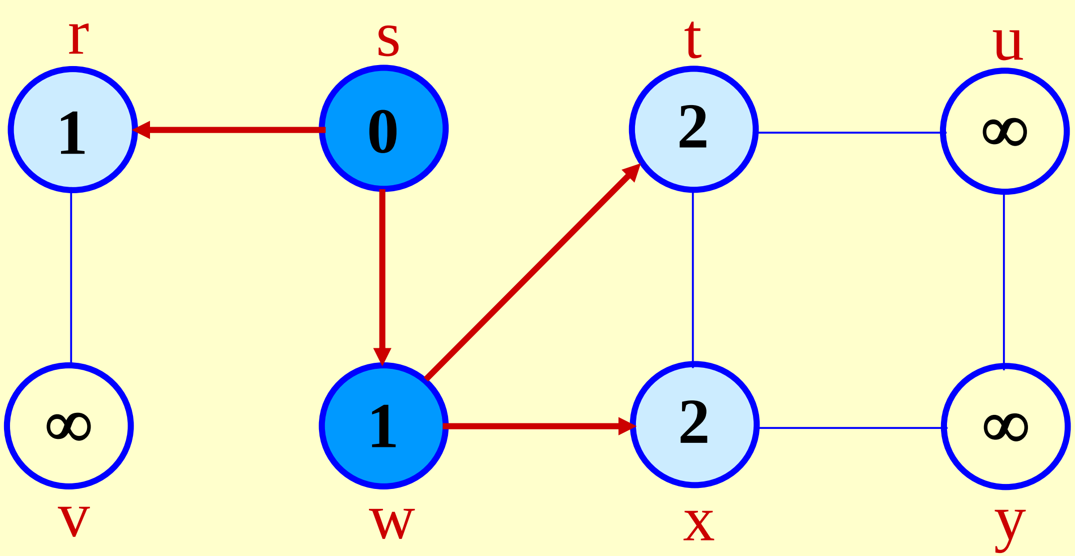

Example (BFS)



Q:	r	t	x
	1	2	2

BFS(G,s)

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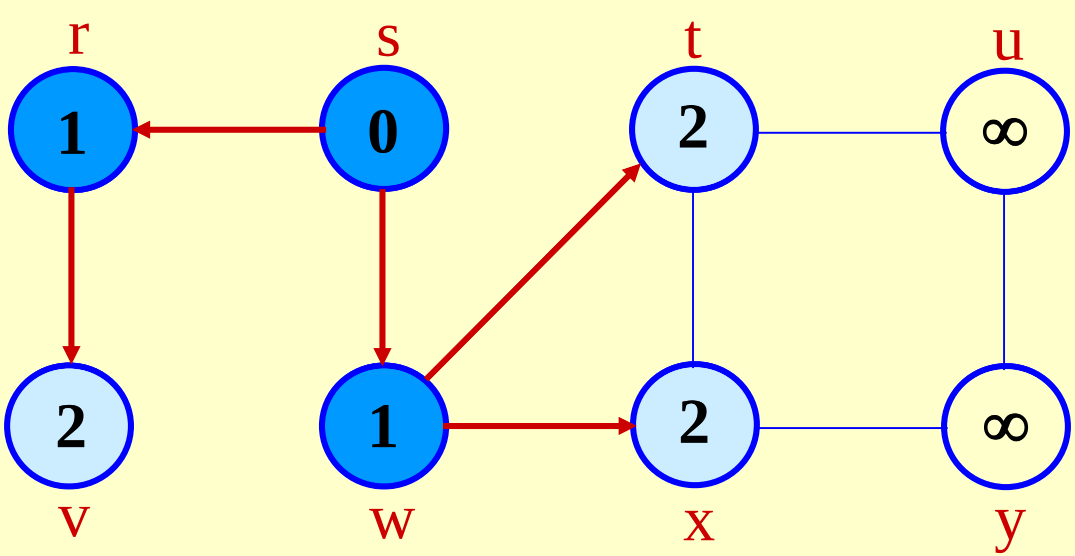

Q: t x v
2 2 2

BFS(G,s)

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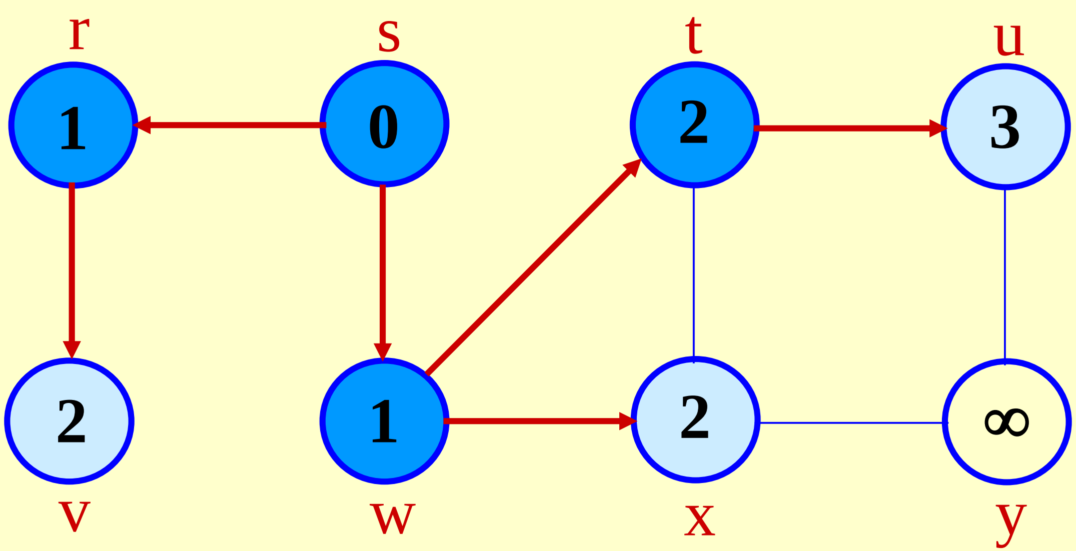
Example (BFS)



Q: x v u
2 2 3

BFS(G,s)

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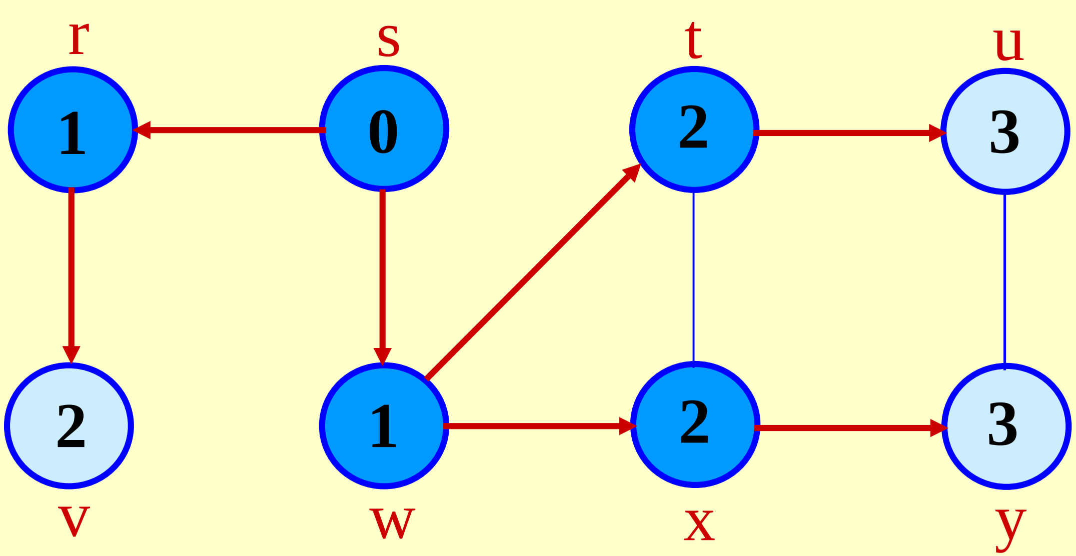
```

Example (BFS)



Q: v u y
2 3 3

BFS(G,s)

```
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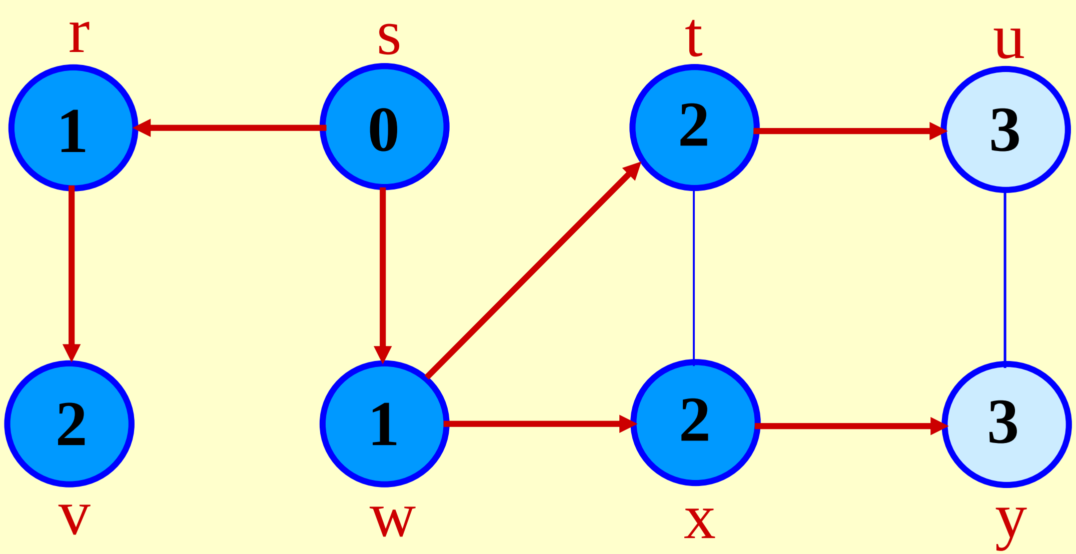
Example (BFS)



Q: u y

3 3

BFS(G,s)

```
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```
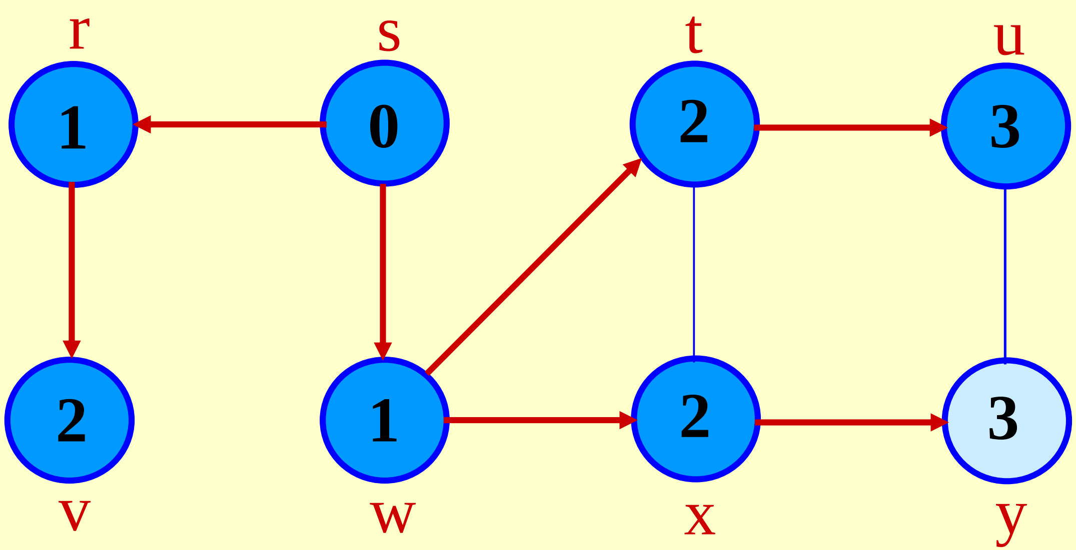

Example (BFS)



Q: y
3

BFS(G,s)

```
for each vertex u in V[G] – {s}
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```
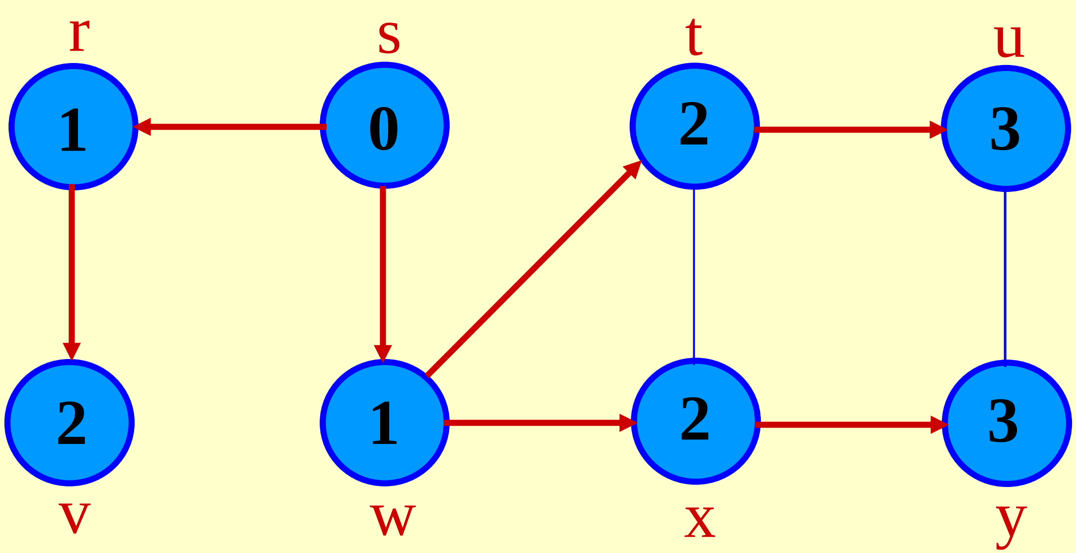

Example (BFS)



Q: ∅

BFS(G,s)

```
for each vertex u in V[G] – {s}
        do color[u] ← white
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           π[u] ← nil
color[s] ← gray
d[s] ← 0
 π[s] ← nil
Q ← Φ
enqueue(Q, s)
while Q ≠ Φ
        do u ← dequeue(Q)
        for each v in Adj[u] do
        if color[v] = white
        then color[v] ← gray
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        π[v] ← u
        enqueue(Q, v)
        color[u] ← black
```
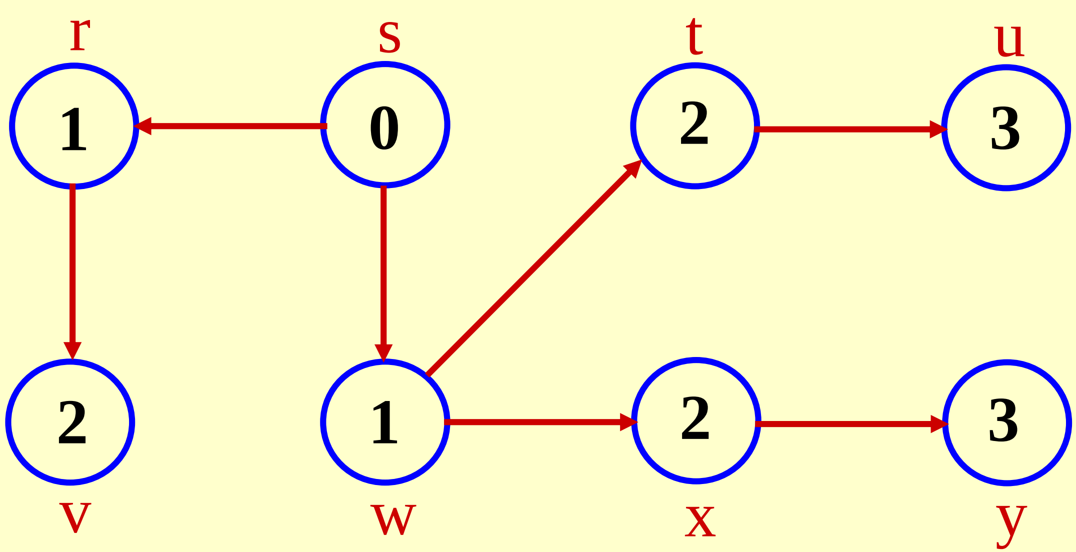

Example (BFS)



BF Tree

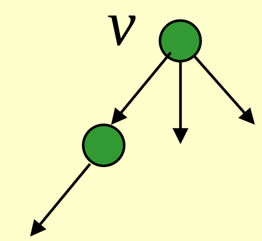
Analysis of BFS

- Initialization takes $O(|V|)$.
- Traversal Loop
  - » After initialization, each vertex is enqueued and dequeued at most once, and each operation takes $O(1)$. So, total time for queuing is $O(|V|)$.
  - » The adjacency list of each vertex is scanned at most once. The sum of lengths of all adjacency lists is $\Theta(|E|)$.
- Summing up over all vertices => total running time of BFS is $O(|V| + |E|)$, linear in the size of the adjacency list representation of graph.
- Correctness Proof
  - » We omit for BFS and DFS.
  - » Will do for later algorithms.

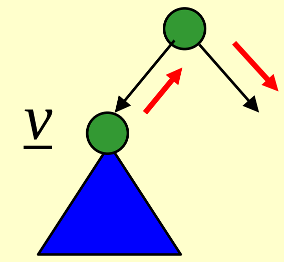
Breadth-first Tree

- For a graph $G = (V, E)$ with source s, the **predecessor subgraph** of G is $G_\pi = (V_\pi, E_\pi)$ where
  - » $V_\pi = \{v \in V : \pi[v] \neq nil\} \cup \{s\}$
  - » $E_\pi = \{(\pi[v], v) \in E : v \in V_\pi - \{s\}\}$
- The predecessor subgraph G_π is a **breadth-first tree** if:
  - » V_π consists of the vertices reachable from s and
  - » for all $v \in V_\pi$, there is a unique simple path from s to v in G_π that is also a shortest path from s to v in G.
- The edges in E_π are called **tree edges**. $|E_\pi| = |V_\pi| - 1$.

Depth-first Search (DFS)



- Explore edges out of the most recently discovered vertex *v*.
- When all edges of *v* have been explored, backtrack to explore other edges leaving the vertex from which *v* was discovered (its *predecessor*).



- "Search as deep as possible first."
- Continue until all vertices reachable from the original source are discovered.
- If any undiscovered vertices remain, then one of them is chosen as a new source and search is repeated from that source.

Depth-first Search

- **Input:** $G = (V, E)$, directed or undirected. No source vertex given!
- **Output:**
  - » 2 **timestamps** on each vertex. Integers between 1 and 2|V|.
    - $d[v]$ = ***discovery time*** (v turns from white to gray)
    - $f[v]$ = ***finishing time*** (v turns from gray to black)
  - » $\pi[v]$: predecessor of $v = u$, such that v was discovered during the scan of u's adjacency list.
- Coloring scheme for vertices as BFS. A vertex is
  - » "undiscovered" (white) when it is not yet encountered.
  - » "discovered" (grey) the first time it is encountered during the search.
  - » "finished" (black) if it is a leaf node or all vertices adjacent to it have been finished.

Pseudo-code

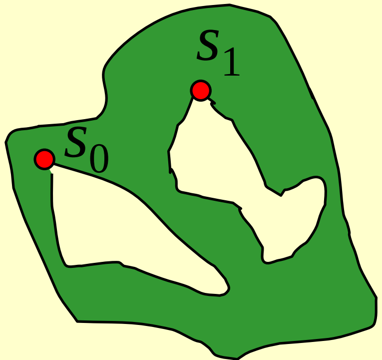
DFS(G)

1. **for** each vertex $u \in V[G]$
2. **do** $\mathit{color}[u] \leftarrow$ white
3. $\pi[u] \leftarrow$ NIL
4. $\mathit{time} \leftarrow 0$
5. **for** each vertex $u \in V[G]$
6. **do if** $\mathit{color}[u] =$ white
7. **then** DFS-Visit(u)

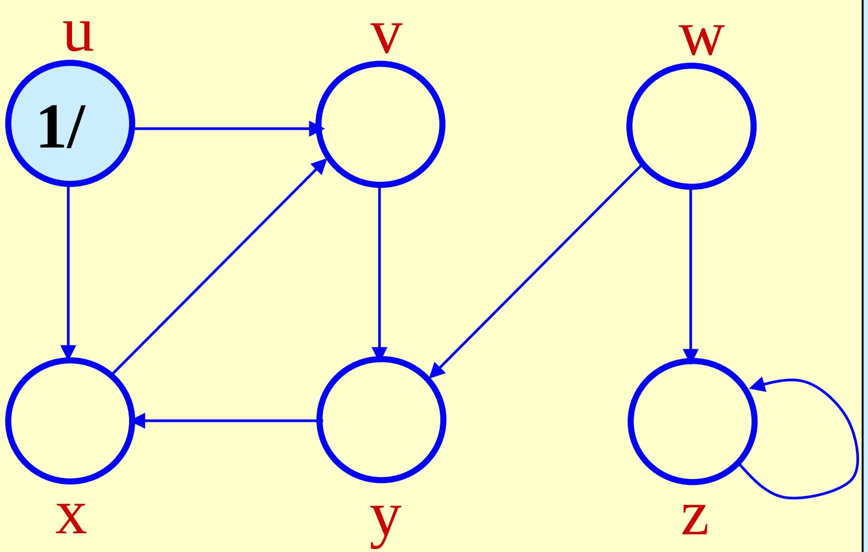
Uses a global timestamp ***time***.

DFS-Visit(u)

1. $\mathit{color}[u] \leftarrow$ GRAY // White vertex u has been discovered
2. $\mathit{time} \leftarrow \mathit{time} + 1$
3. $d[u] \leftarrow \mathit{time}$
4. **for** each $v \in \mathit{Adj}[u]$
5. **do if** $\mathit{color}[v] =$ WHITE
6. **then** $\pi[v] \leftarrow u$
7. DFS-Visit(v)
8. $\mathit{color}[u] \leftarrow$ BLACK // Blacken u; it is finished.
9. $f[u] \leftarrow \mathit{time} \leftarrow \mathit{time} + 1$



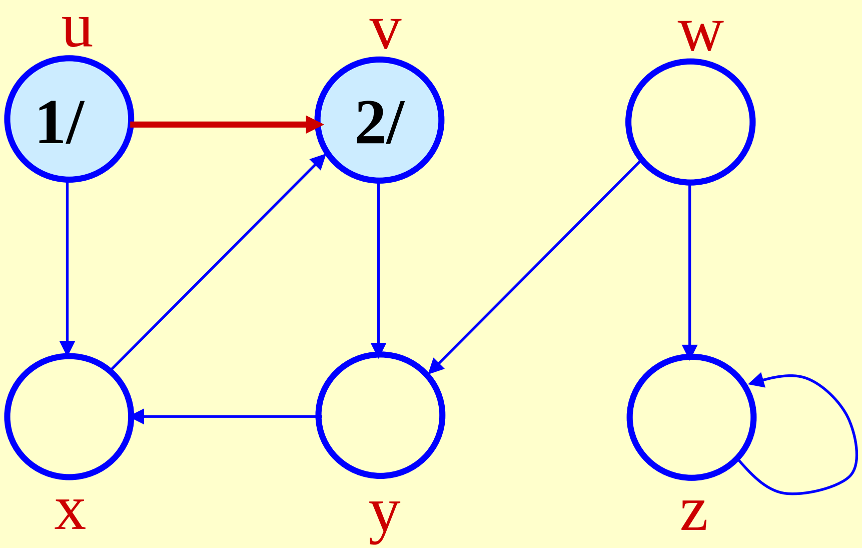
Example (DFS)



DFS-Visit(*u*)

1. *color*[*u*] ← GRAY // White vertex *u* has been discovered
2. *time* ← *time* + 1
3. *d*[*u*] ← *time*
4. **for** each *v* ∈ *Adj*[*u*]
5. **do if** *color*[*v*] = WHITE
6. **then** π[*v*] ← *u*
7. DFS-Visit(*v*)
8. *color*[*u*] ← BLACK // Blacken *u*; it is finished.
9. *f*[*u*] ← *time* ← *time* + 1

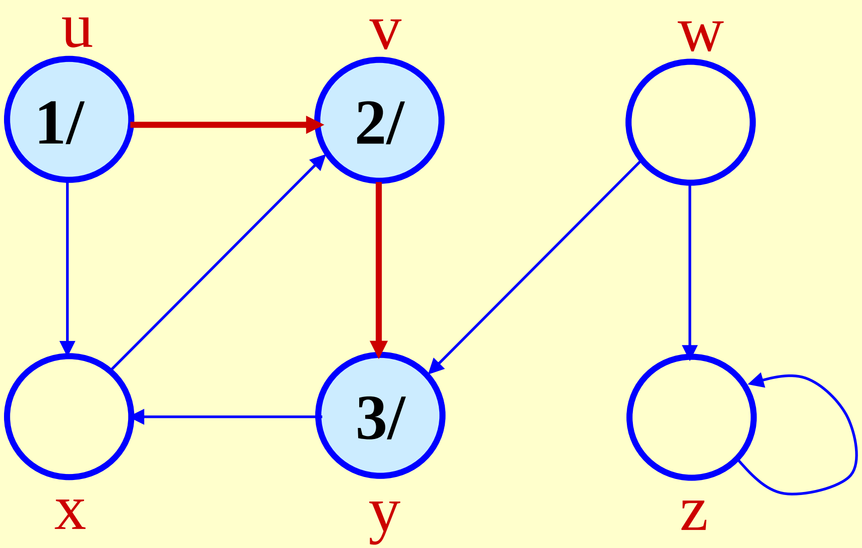
Example (DFS)



DFS-Visit(u)

1. *color*[u] $\leftarrow$ GRAY // White vertex u has been discovered
2. *time* $\leftarrow$ *time* + 1
3. $d[u] \leftarrow$ *time*
4. **for** each $v \in Adj[u]$
5. **do if** *color*[v] = WHITE
6. **then** $\pi[v] \leftarrow u$
7. DFS-Visit(v)
8. *color*[u] $\leftarrow$ BLACK // Blacken u; it is finished.
9. $f[u] \leftarrow$ *time* $\leftarrow$ *time* + 1

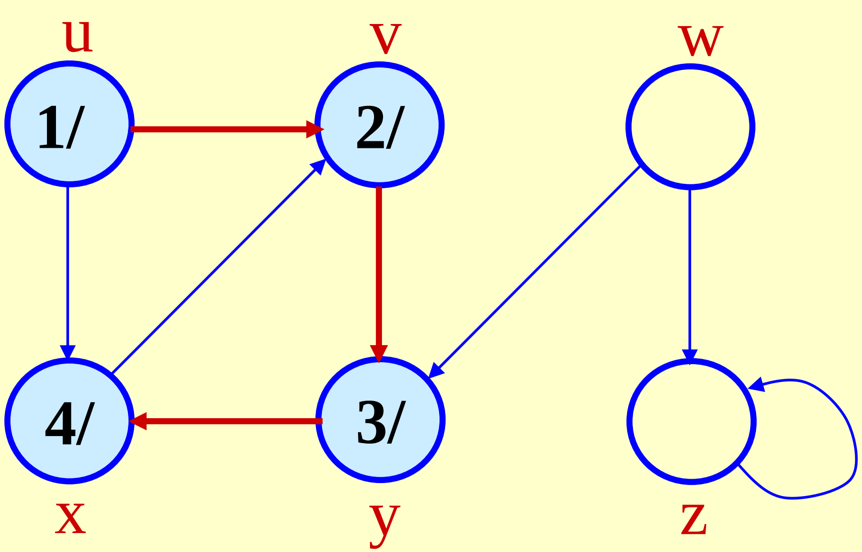
Example (DFS)



DFS-Visit(*u*)

1. *color*[*u*] ← GRAY // White vertex *u* has been discovered
2. *time* ← *time* + 1
3. *d*[*u*] ← *time*
4. **for** each *v* ∈ *Adj*[*u*]
5. **do if** *color*[*v*] = WHITE
6. **then** π[*v*] ← *u*
7. DFS-Visit(*v*)
8. *color*[*u*] ← BLACK // Blacken *u*; it is finished.
9. *f*[*u*] ← *time* ← *time* + 1

Example (DFS)



DFS-Visit(u)

color[u] $\leftarrow$ GRAY // White vertex u has been discovered

time $\leftarrow$ *time* + 1

$d[u] \leftarrow$ *time*

for each $v \in Adj[u]$

  do if *color*[v] = WHITE

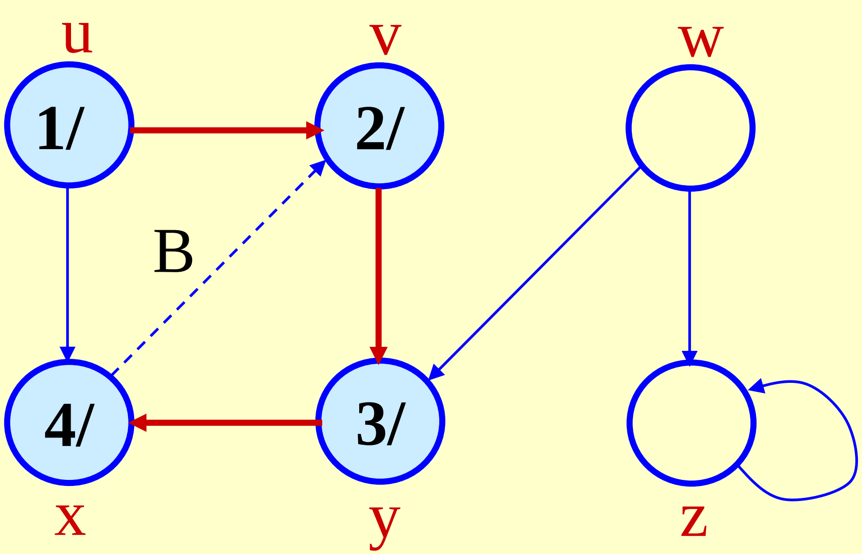
    then $\pi[v] \leftarrow u$

      DFS-Visit(v)

color[u] $\leftarrow$ BLACK // Blacken u; it is finished.

$f[u] \leftarrow$ *time* $\leftarrow$ *time* + 1

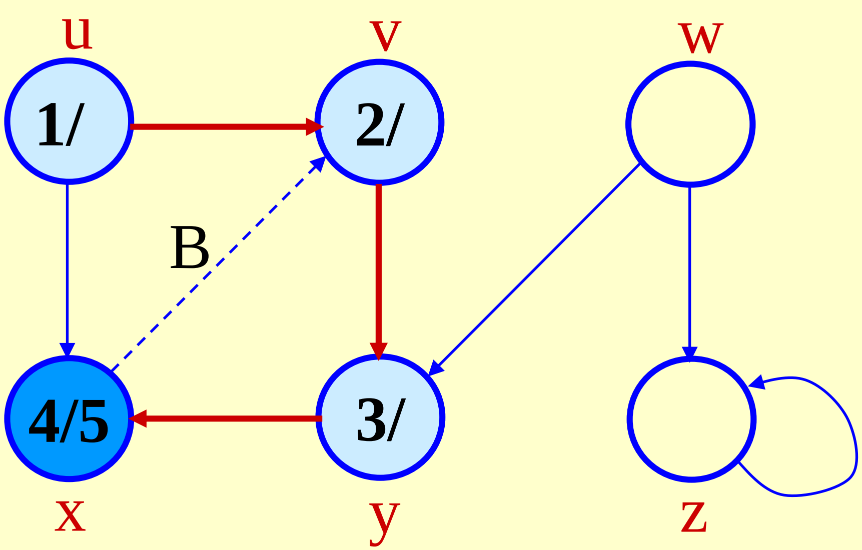
Example (DFS)



DFS-Visit(*u*)

1. *color*[*u*] ← GRAY // White vertex *u* has been discovered
2. *time* ← *time* + 1
3. *d*[*u*] ← *time*
4. **for** each *v* ∈ *Adj*[*u*]
5. **do if** *color*[*v*] = WHITE
6. **then** π[*v*] ← *u*
7. DFS-Visit(*v*)
8. *color*[*u*] ← BLACK // Blacken *u*; it is finished.
9. *f*[*u*] ← *time* ← *time* + 1

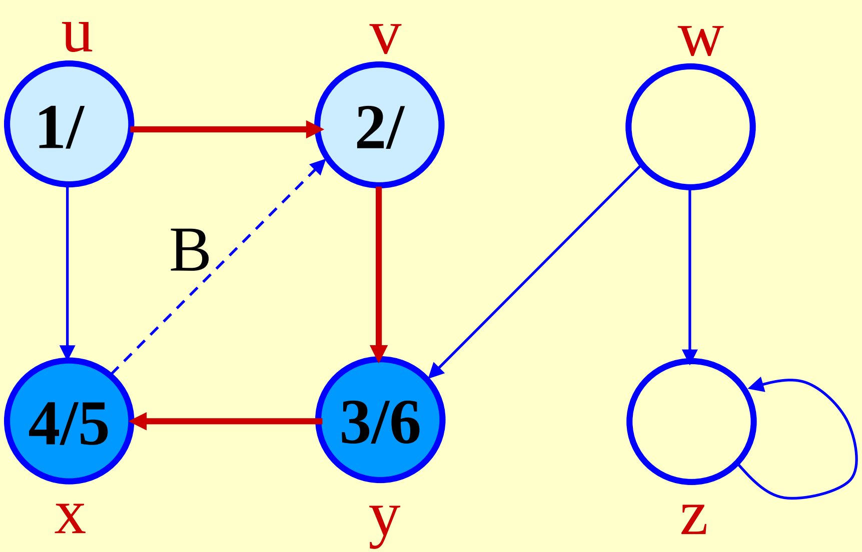
Example (DFS)



DFS-Visit(*u*)

1. *color*[*u*] ← GRAY // White vertex *u* has been discovered
2. *time* ← *time* + 1
3. *d*[*u*] ← *time*
4. **for** each *v* ∈ *Adj*[*u*]
5. **do if** *color*[*v*] = WHITE
6. **then** π[*v*] ← *u*
7. DFS-Visit(*v*)
8. *color*[*u*] ← BLACK // Blacken *u*; it is finished.
9. *f*[*u*] ← *time* ← *time* + 1

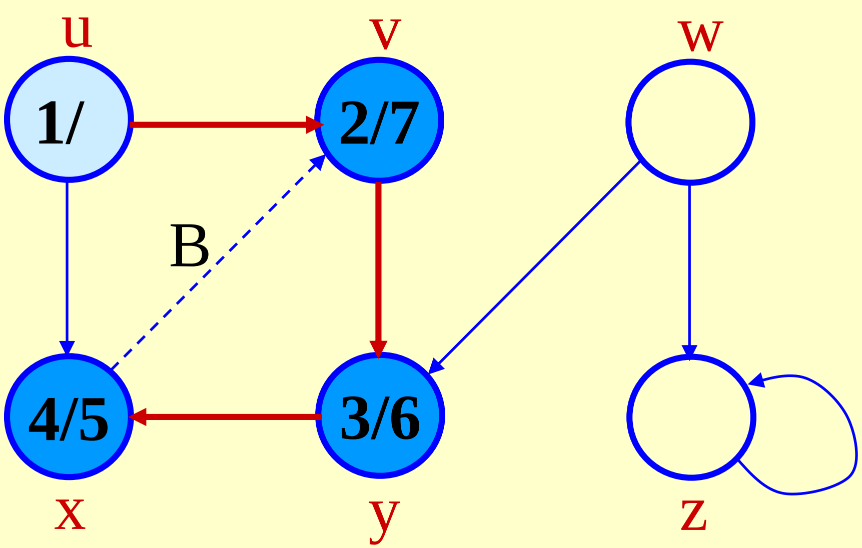
Example (DFS)



DFS-Visit(*u*)

1. *color*[*u*] ← GRAY // White vertex *u* has been discovered
2. *time* ← *time* + 1
3. *d*[*u*] ← *time*
4. **for** each *v* ∈ *Adj*[*u*]
5. **do if** *color*[*v*] = WHITE
6. **then** π[*v*] ← *u*
7. DFS-Visit(*v*)
8. *color*[*u*] ← BLACK // Blacken *u*; it is finished.
9. *f*[*u*] ← *time* ← *time* + 1

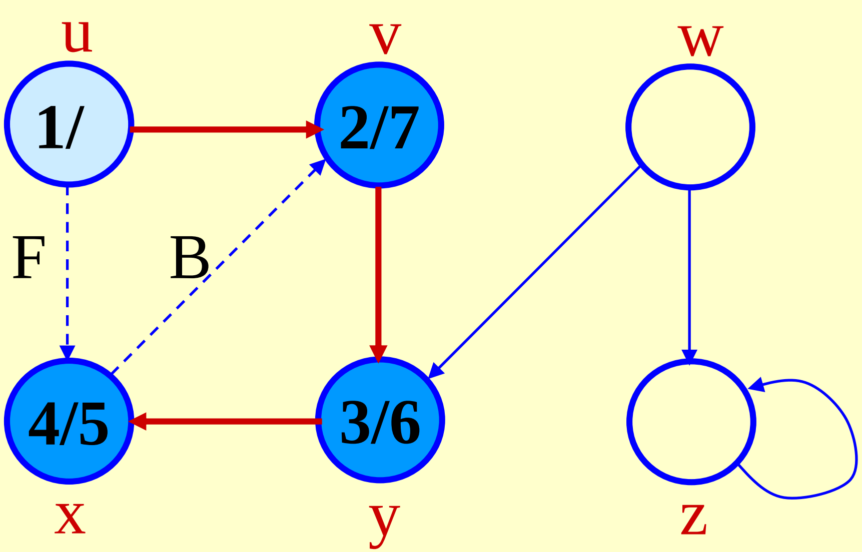
Example (DFS)



DFS-Visit(*u*)

```
color[u] ← GRAY  // White vertex u
                    has been discovered
time ← time + 1
 d[u] ← time
 for each v ∈ Adj[u]
       do if color[v] = WHITE
              then π[v] ← u
                     DFS-Visit(v)
  color[u] ← BLACK     // Blacken u;
                          it is finished.
  f[u] ← time ← time + 1
```

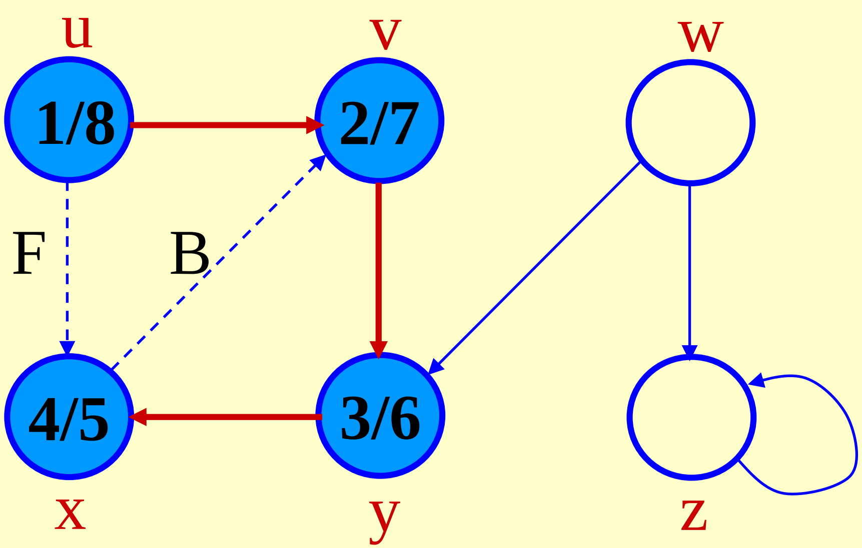
Example (DFS)



DFS-Visit(u)

1. *color*[u] $\leftarrow$ GRAY // White vertex u has been discovered
2. $time \leftarrow time + 1$
3. $d[u] \leftarrow time$
4. **for** each $v \in Adj[u]$
5. **do if** *color*[v] = WHITE
6. **then** $\pi[v] \leftarrow u$
7. DFS-Visit(v)
8. *color*[u] $\leftarrow$ BLACK // Blacken u; it is finished.
9. $f[u] \leftarrow time \leftarrow time + 1$

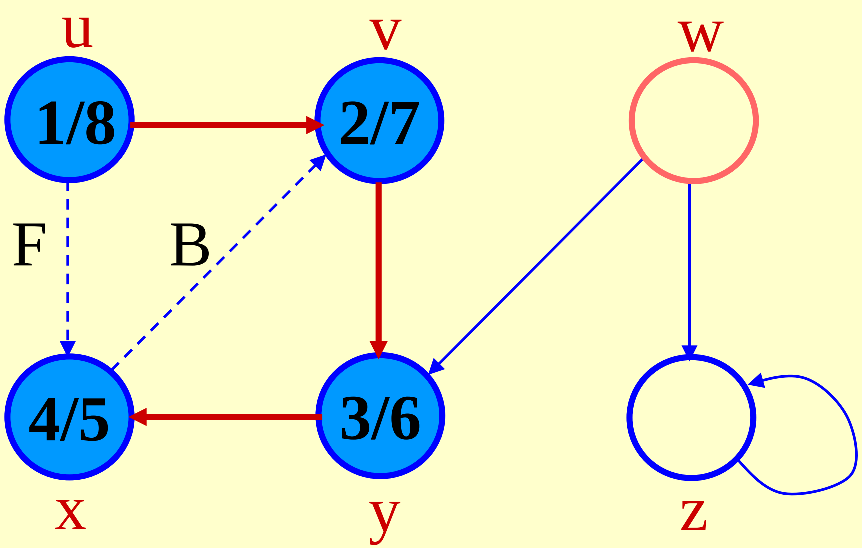
Example (DFS)



DFS-Visit(*u*)

color[*u*] ← GRAY // White vertex *u* has been discovered
time ← *time* + 1
d[*u*] ← *time*
for each *v* ∈ *Adj*[*u*]
  do if *color*[*v*] = WHITE
    then π[*v*] ← *u*
      DFS-Visit(*v*)
color[*u*] ← BLACK // Blacken *u*; it is finished.
f[*u*] ← *time* ← *time* + 1

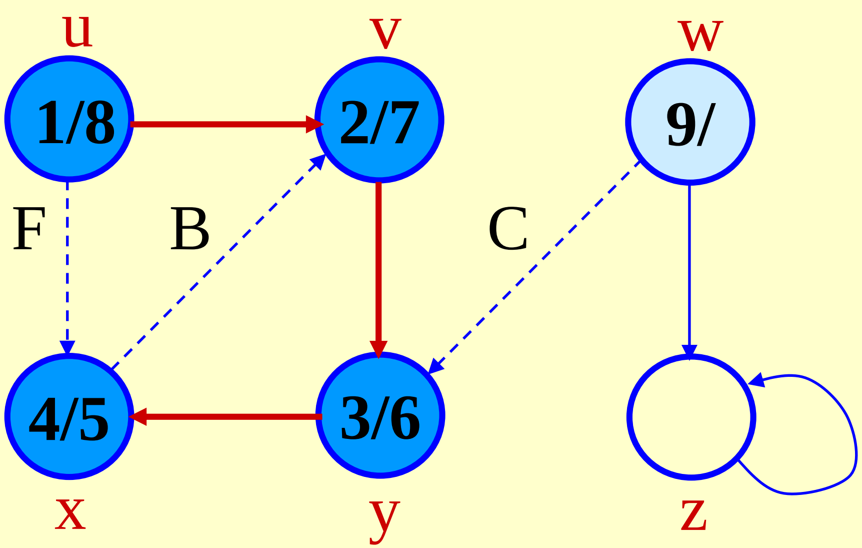
Example (DFS)



DFS(*G*)

1. **for** each vertex $u \in V[G]$
2. **do** $color[u] \leftarrow$ white
3. $\pi[u] \leftarrow$ NIL
4. $time \leftarrow 0$
5. **for** each vertex $u \in V[G]$
6. **do if** $color[u] =$ white
7. **then** DFS-Visit(u)

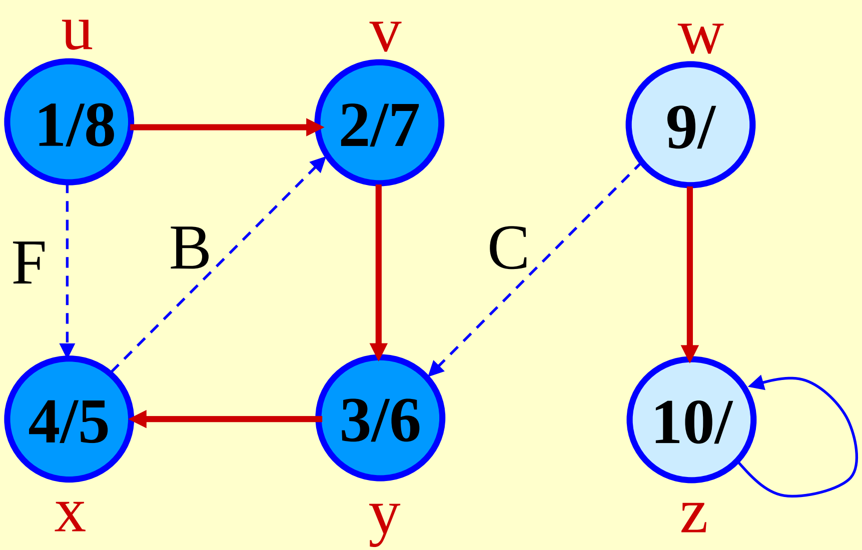
Example (DFS)



DFS-Visit(u)

1. *color*[u] $\leftarrow$ GRAY // White vertex u has been discovered
2. *time* $\leftarrow$ *time* + 1
3. $d[u] \leftarrow$ *time*
4. **for** each $v \in Adj[u]$
5. **do if** *color*[v] = WHITE
6. **then** $\pi[v] \leftarrow u$
7. DFS-Visit(v)
8. *color*[u] $\leftarrow$ BLACK // Blacken u; it is finished.
9. $f[u] \leftarrow$ *time* $\leftarrow$ *time* + 1

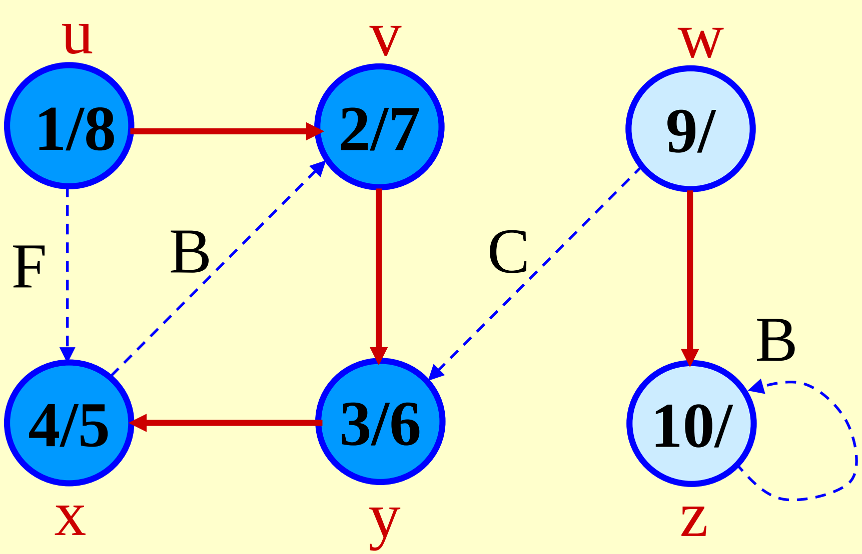
Example (DFS)



<u>DFS-Visit(*u*)</u>

1. *color*[*u*] ← GRAY // White vertex *u* has been discovered
2. *time* ← *time* + 1
3. *d*[*u*] ← *time*
4. **for** each $v \in Adj[u]$
5. **do if** *color*[*v*] = WHITE
6. **then** $\pi[v] \leftarrow u$
7. DFS-Visit(*v*)
8. *color*[*u*] ← BLACK // Blacken *u*; it is finished.
9. *f*[*u*] ← *time* ← *time* + 1

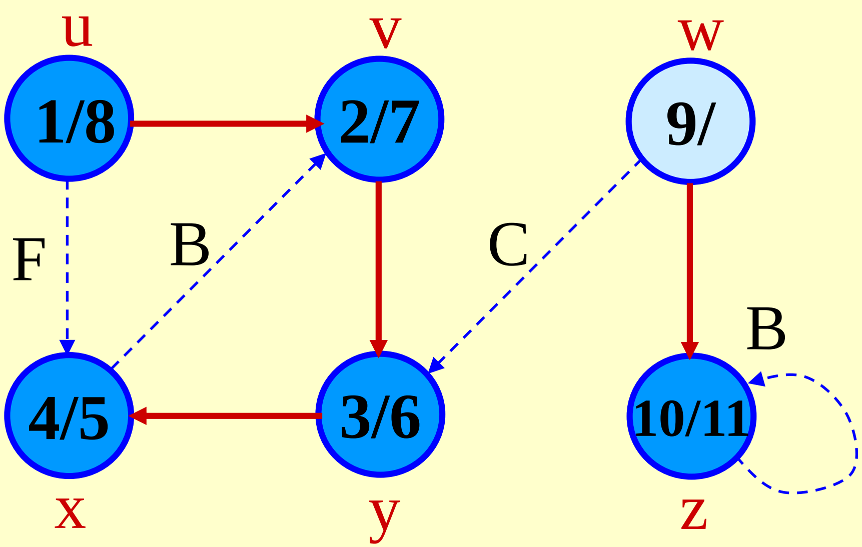
Example (DFS)



DFS-Visit(u)

1. *color*[u] ← GRAY // White vertex u has been discovered
2. *time* ← *time* + 1
3. $d[u]$ ← *time*
4. **for** each $v \in Adj[u]$
5. **do if** *color*[v] = WHITE
6. **then** $\pi[v] \leftarrow u$
7. DFS-Visit(v)
8. *color*[u] ← BLACK // Blacken u; it is finished.
9. $f[u] \leftarrow time \leftarrow time + 1$

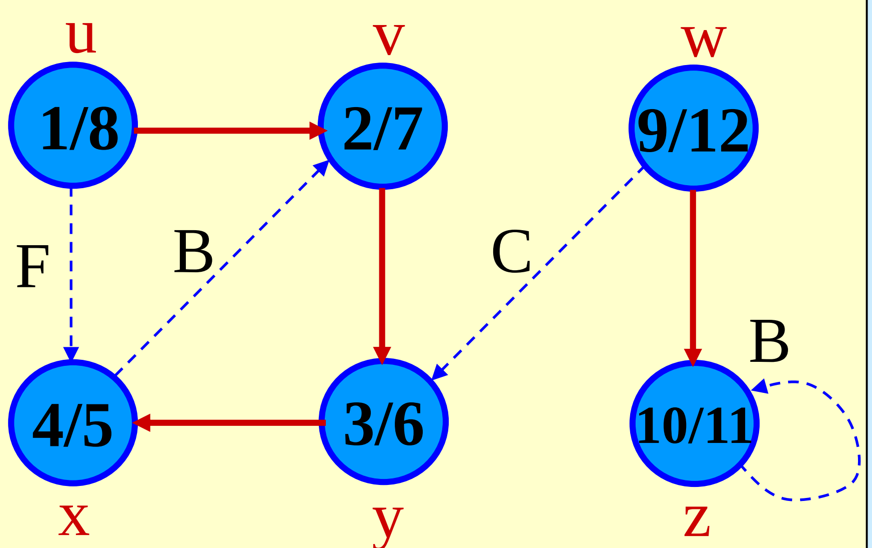
Example (DFS)



DFS-Visit(u)

1. *color*[u] ← GRAY // White vertex u has been discovered
2. *time* ← *time* + 1
3. $d[u]$ ← *time*
4. **for** each $v \in Adj[u]$
5. **do if** *color*[v] = WHITE
6. **then** $\pi[v] \leftarrow u$
7. DFS-Visit(v)
8. *color*[u] ← BLACK // Blacken u; it is finished.
9. $f[u] \leftarrow time \leftarrow time + 1$

Example (DFS)



<u>DFS-Visit(*u*)</u>

1. *color*[*u*] ← GRAY // White vertex *u* has been discovered
2. *time* ← *time* + 1
3. *d*[*u*] ← *time*
4. **for** each *v* ∈ *Adj*[*u*]
5. **do if** *color*[*v*] = WHITE
6. **then** π[*v*] ← *u*
7. DFS-Visit(*v*)
8. *color*[*u*] ← BLACK // Blacken *u*; it is finished.
9. *f*[*u*] ← *time* ← *time* + 1

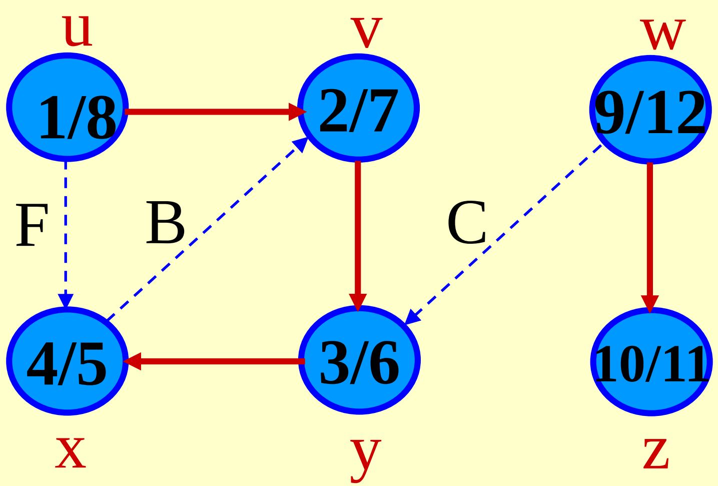
Analysis of DFS

- Loops on lines 1-2 & 5-7 take $\Theta(|V|)$ time, excluding time to execute DFS-Visit.
- DFS-Visit is called once for each white vertex $v \in V$ when it's painted gray the first time. Lines 3-6 of DFS-Visit is executed |Adj[v]| times. The total cost of executing DFS-Visit is $\sum_{v \in V} |\mathrm{Adj}[v]| = \Theta(|E|)$
- Total running time of DFS is $\Theta(|V| + |E|)$.

Depth-First Trees

- Predecessor subgraph defined slightly different from that of BFS.
- The predecessor subgraph of DFS is $G_\pi = (V, E_\pi)$ where $E_\pi = \{(\pi[v], v) : v \in V \text{ and } \pi[v] \neq nil\}$.
  - » How does it differ from that of BFS?
  - » The predecessor subgraph G_π forms a *depth-first forest* composed of several *depth-first trees*. The edges in E_π are called *tree edges*.

Definition:
Forest: An acyclic graph G that may be disconnected.



Parenthesis Theorem

Theorem 22.7

For all u, v, exactly one of the following holds:

1. $d[u] < f[u] < d[v] < f[v]$ or $d[v] < f[v] < d[u] < f[u]$ and neither u nor v is a descendant of the other in the *DF-tree*.
2. $d[u] < d[v] < f[v] < f[u]$ and v is a descendant of u in *DF-tree*.
3. $d[v] < d[u] < f[u] < f[v]$ and u is a descendant of v in *DF-tree*.

- So $d[u] < d[v] < f[u] < f[v]$ *cannot* happen.
- Like parentheses:
  - OK: ( ) [ ] ([ ]) [( )]
  - Not OK: ([ )] [( ])

$d[v]$ $f[v]$

([ )]

$d[u]$ $f[u]$

( )

$d[v]$ $f[v]$

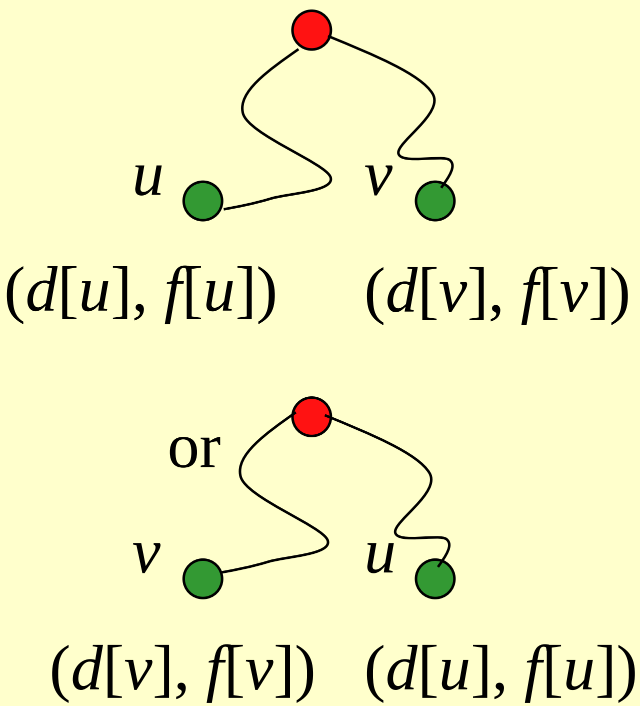
Corollary

([ ])

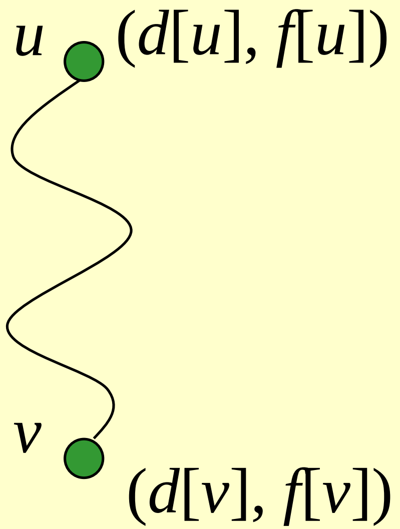
v is a proper descendant of u if and only if $d[u] < d[v] < f[v] < f[u]$.

Parenthesis Theorem

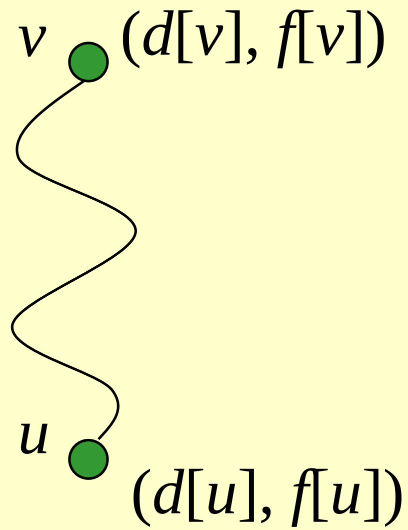
Case 1:



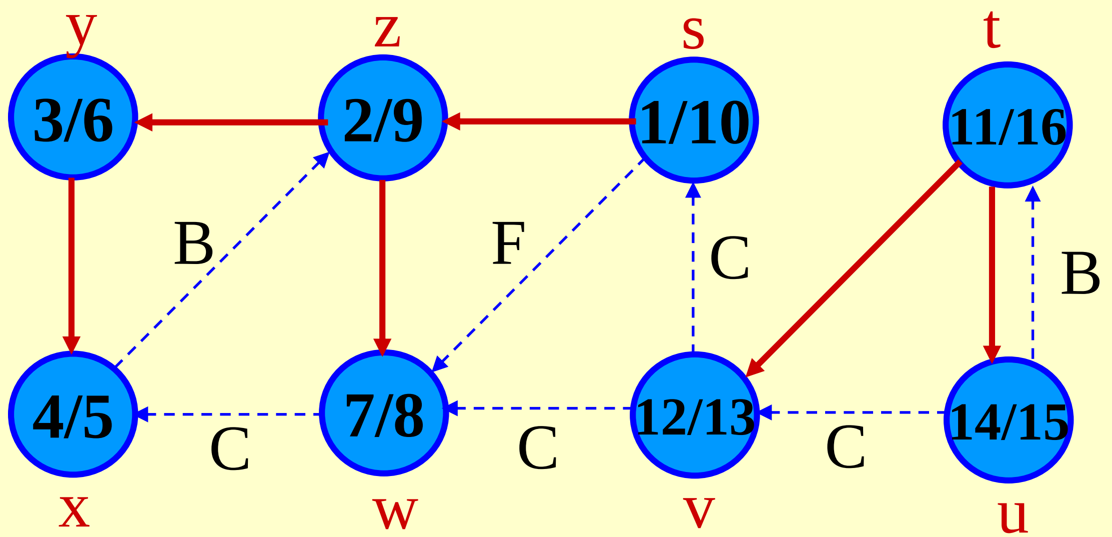
Case 2:



Case 3:



Example (Parenthesis Theorem)



(s (z (y (x x) y) (w w) z) s) (t (v v) (u u) t)

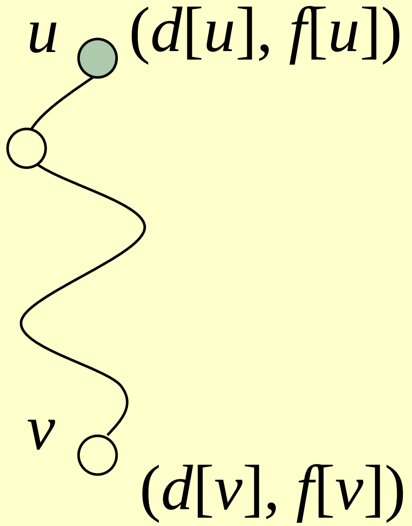
1<2<3<4<5<6<7<8<9<10 11<12<13<14<15<16

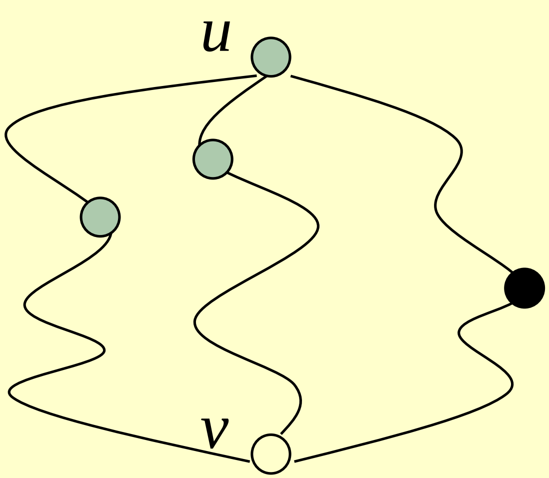
In general, if we use ‘(v’ to represent $d[v]$, and ‘v)’ to represent $f[v]$, the inequalities in the Parenthesis Theorem are just like parentheses in an arithmetical expression.

White-path Theorem

Theorem 22.9

v is a descendant of u in *DF-tree* if and only if at time $d[u]$, there is a path $u \rightsquigarrow v$ consisting of only white vertices. (Except for u, which was *just* colored gray.)



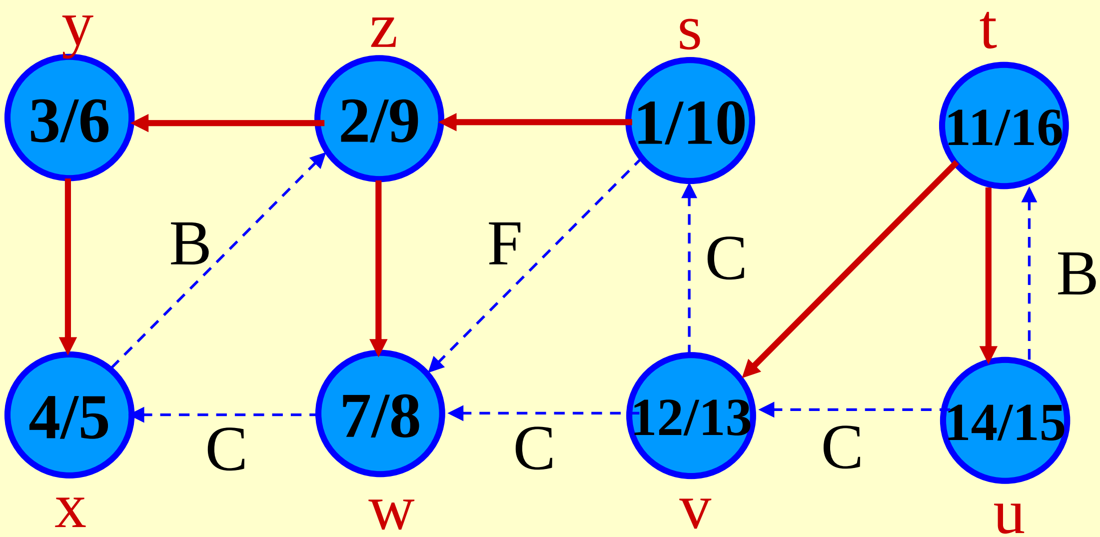


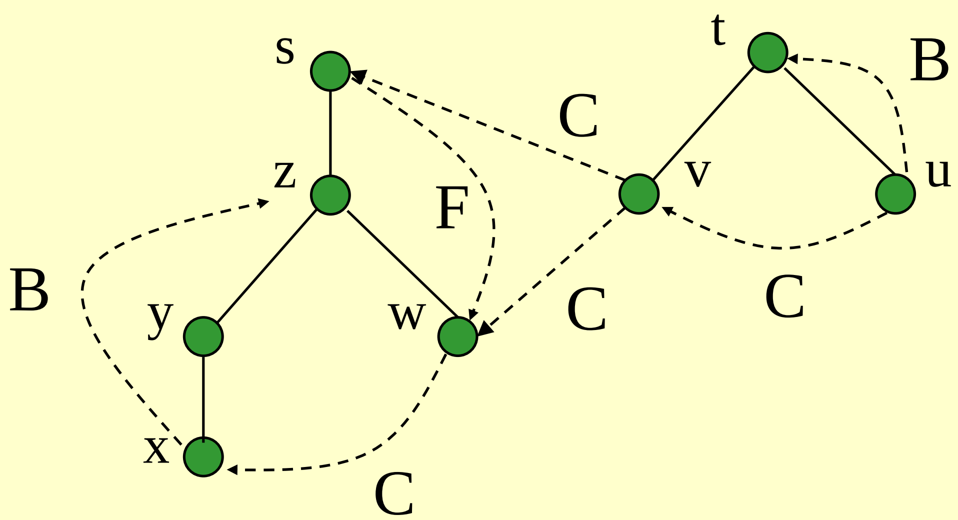
Classification of Edges

- **Tree edge:** in the depth-first forest. Found by exploring (u, v).
- **Back edge:** (u, v), where u is a descendant of v (in the depth-first tree).
- **Forward edge:** (u, v), where v is a descendant of u, but not a tree edge.
- **Cross edge:** any other edge (u, v) such that u is not a descendant of v (in the depth-first tree) and *vice versa*.

Theorem:
In DFS of an undirected graph, we get only tree and back edges. No forward or cross edges.

Classification of Edges



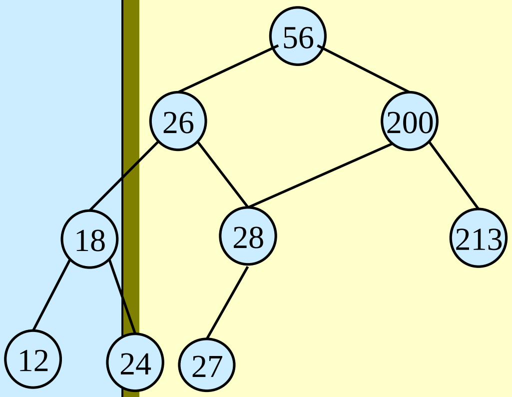


DFS graph search using stack

<u>Depth-first(x)</u>

1. **push**(S, x)
2. **while** $S \neq$ empty **do**
3. v := **pop**(S)
4. print $key[x]$
5. let $v_1, \ldots, v_k$ be the children of x
6. **for** ($i = k$ to 1) **do**
7. **if** v_i has not yet been accessed **then**
8. **push**(S, v_i)

S is a stack.



It is also called the preoreder search and top-down search.

Bottom-up search of a directed graph

Bottom-up(x)

Bottom-up(x)

1.let $v_1, \ldots, v_k$ be the children of x

2.for ($i = k$ to 1) **do**

3. **if** v_i has not yet accessed **then**

4. **Bottom-up**(v_i)

5. Print(x)