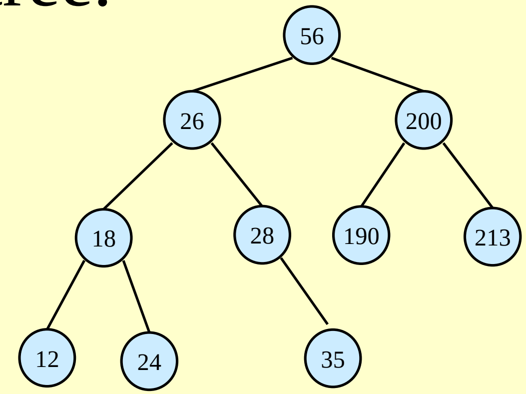
Binary Search Trees

- What is a binary search tree?
- Tree searching
- Inorder traversal of a binary search tree
- Find Min & Max
- Predecessor and successor
- BST insertion and deletion

Binary Trees

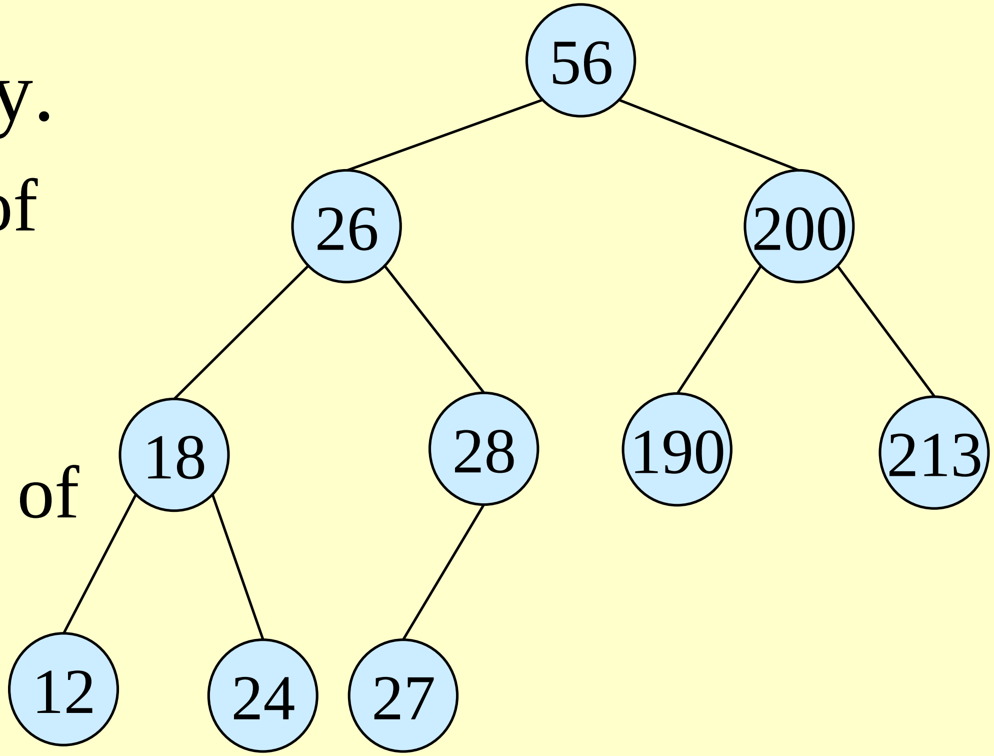
- Recursive definition
  1. An empty tree is a binary tree
  2. A node with two child subtrees is a binary tree
  3. Let *A* and *B* be two binary trees. A tree with root *r*, and *A* and *B* as its left and right subtrees, respectively, is a binary tree.

Is this a binary tree?



Binary Search Tree

- Stored keys must satisfy the *binary search tree* property.
  - » $\forall$ *y* in left subtree of *x*, then *key*[*y*] < *key*[*x*].
  - » $\forall$ *y* in right subtree of *x*, then *key*[*y*] $\geq$ *key*[*x*].



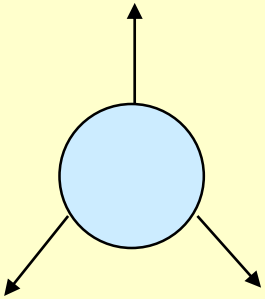
Binary Search Trees

- A BST is a data structures that can support dynamic set operations.
  - » Search, Inorder traversal, Minimum, Maximum, Predecessor, Successor, Insert, and Delete.
- Can be used to build
  - » Dictionaries.
  - » Priority Queues.
- Basic operations take time proportional to the height of the tree – $O(h)$.

BST – Representation

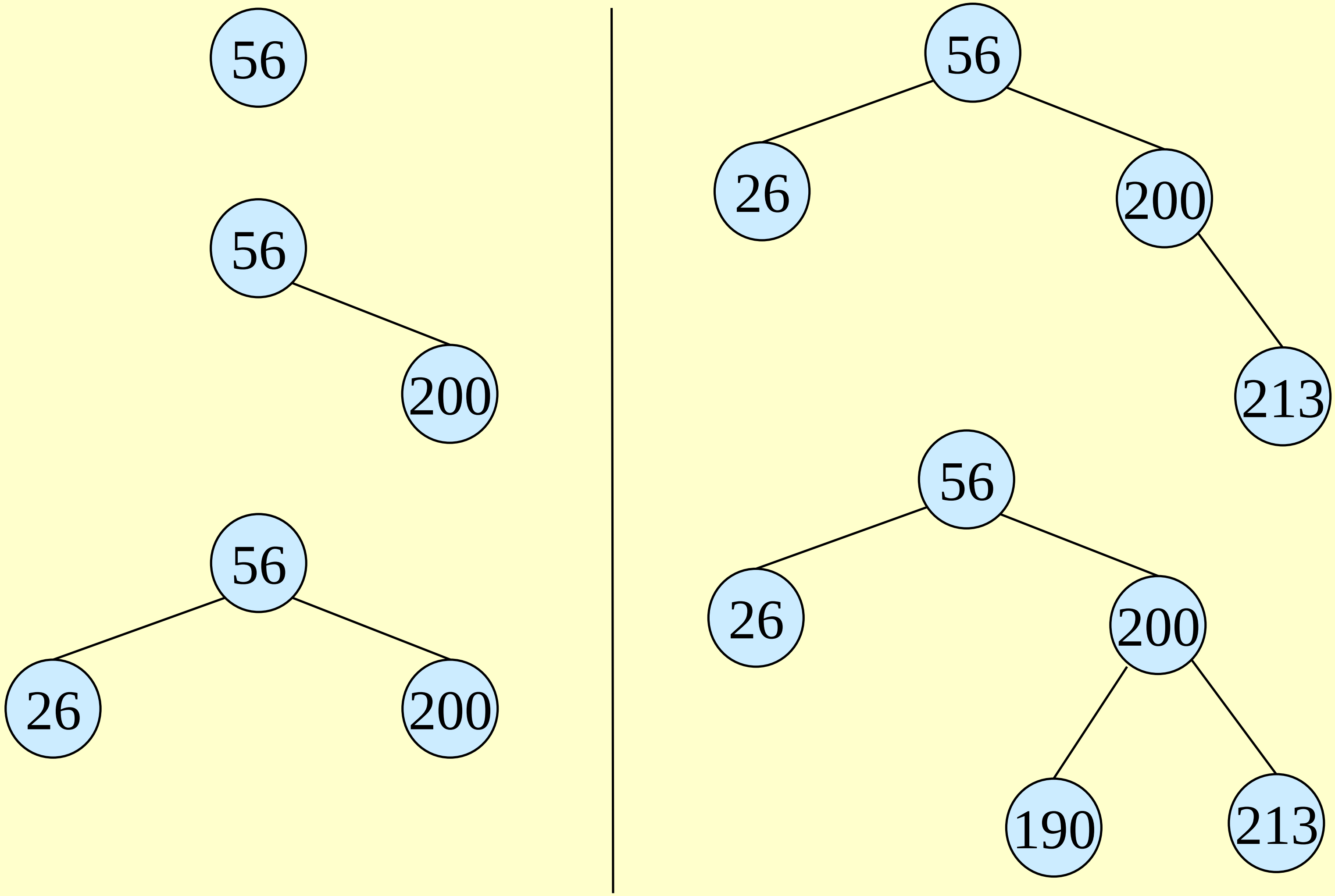
- Represented by a linked data structure of nodes.
- *root*(*T*) points to the root of tree *T*.
- Each node contains fields:
  - » *key*
  - » *left* – pointer to left child: root of left subtree.
  - » *right* – pointer to right child : root of right subtree.
  - » *p* – pointer to parent. *p*[*root*[*T*]] = *NIL* (optional).

```
Class Node {
        key     string;
        left    Node;
        right   Node;
        p       Node;
}
```



Binary Search Tree Construction

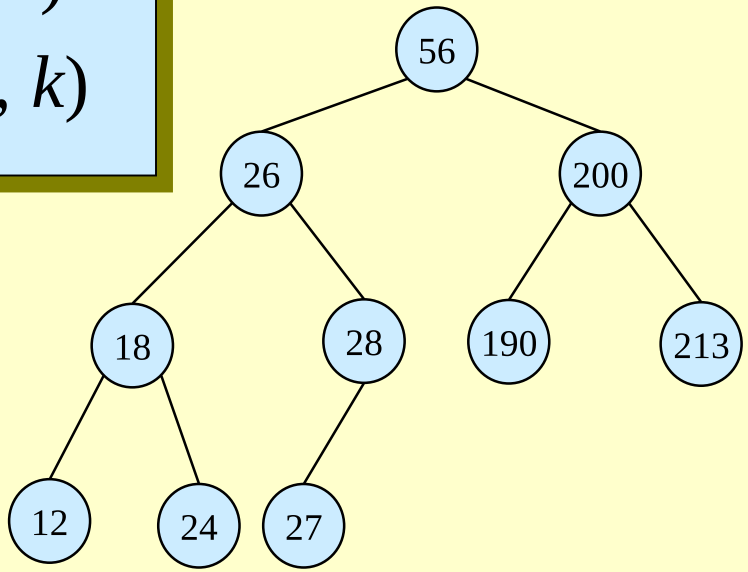
56, 200, 26, 213, 190, 28, 27, 18, 12, 24



Tree Search

Tree-Search(x, k)

1. **if** x = NIL *or* k = *key*[x]
2. **then** return x
3. **if** k < *key*[x]
4. **then** return Tree-Search(*left*[x], k)
5. **else** return Tree-Search(*right*[x], k)



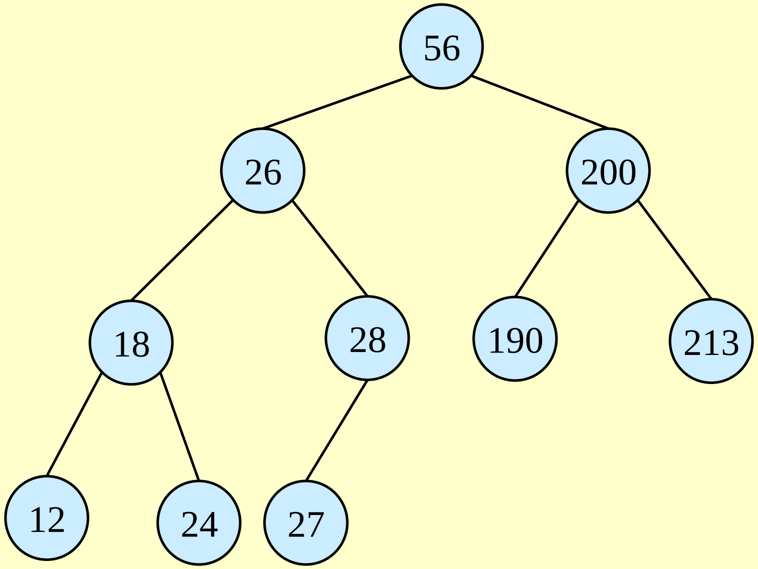
Running time: *O*(*h*)

Aside: tail-recursion

Iterative Tree Search

Iterative-Tree-Search(x, k)

while $x \neq NIL$ **and** $k \neq key[x]$
    do if $k < key[x]$
        then $x \leftarrow left[x]$
        else $x \leftarrow right[x]$
return x



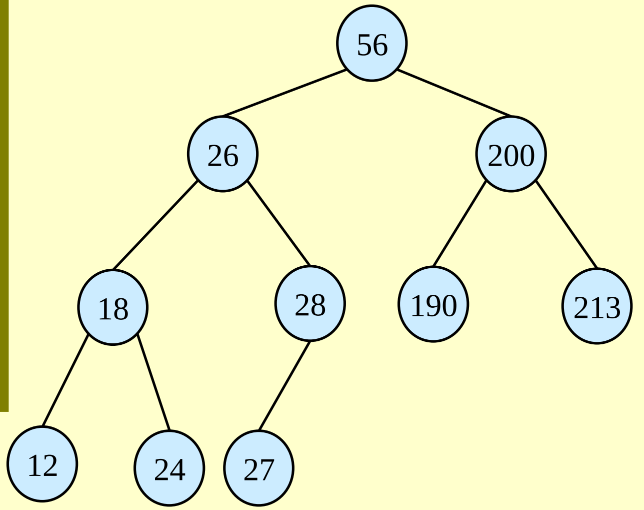
The iterative tree search is more efficient on most computers.
The recursive tree search is more straightforward.

Inorder Traversal

The binary-search-tree property allows the keys of a binary search tree to be printed, in (monotonically increasing) order, recursively.

<u>Inorder-Tree-Walk (x)</u>

1. **if** $x \neq$ NIL
2. **then** Inorder-Tree-Walk(*left*[x])
3. print *key*[x]
4. Inorder-Tree-Walk(*right*[x])



- How long does the walk take?
- Can you prove its correctness?

Correctness of Inorder-Walk

- Must prove that it prints all elements, in order, and that it terminates.
- By induction on size of tree. Size=0: Easy.
- Size >1:
  - » Prints left subtree in order by induction.
  - » Prints root, which comes after all elements in left subtree (still in order).
  - » Prints right subtree in order (all elements come after root, so still in order).

Querying a Binary Search Tree

- All dynamic-set search operations can be supported in $O(h)$ time.
- $h = \Theta(lg\ n)$ for a balanced binary tree (and for an average tree built by adding nodes in random order.)
- $h = \Theta(n)$ for an unbalanced tree that resembles a linear chain of n nodes in the worst case.

Exercise: Sorting Using BSTs

Sort (A)

  for $i \leftarrow 1$ to n

    do tree-insert($A[i]$)

  inorder-tree-walk(*root*)

- » What are the worst case and best case running times?
- » In practice, how would this compare to other sorting algorithms?

Finding Min & Max

- The binary-search-tree property guarantees that:
  - » The minimum is located at the left-most node.
  - » The maximum is located at the right-most node.

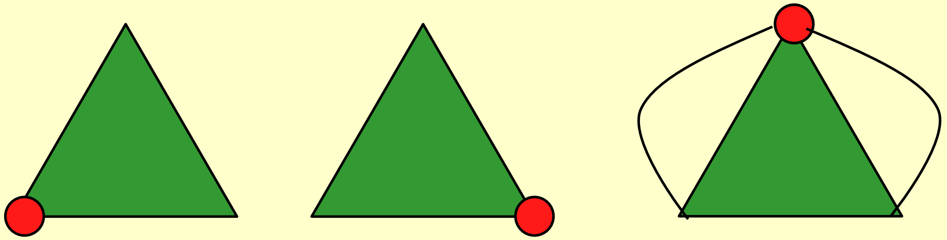
Tree-Minimum(x)

1. **while** $left[x] \neq NIL$
2. **do** $x \leftarrow left[x]$
3. **return** x

Tree-Maximum(x)

1. **while** $right[x] \neq NIL$
2. **do** $x \leftarrow right[x]$
3. **return** x

Q: How long do they take?

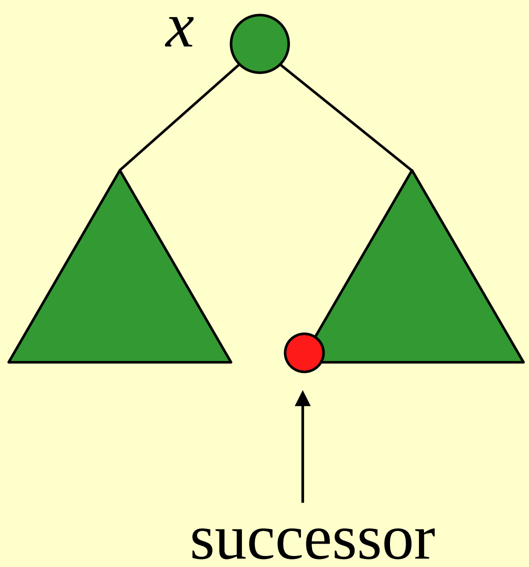


Predecessor and Successor

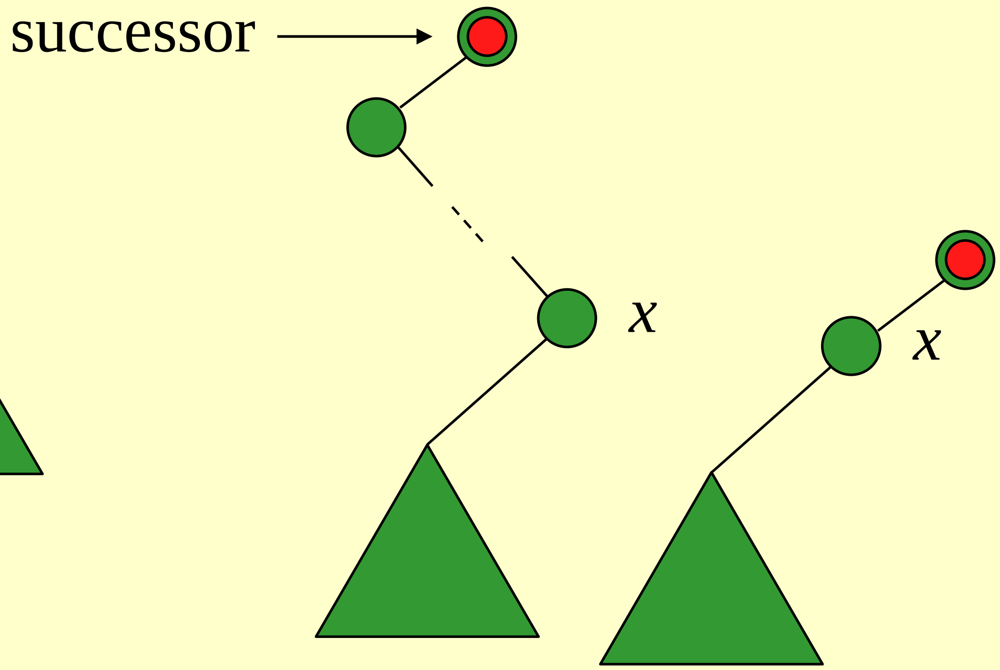
- Predecessor of node *x* is the node *y* such that *key*[*y*] is the greatest key smaller than *key*[*x*].
- Successor of node *x* is the node *y* such that *key*[*y*] is the smallest key greater than *key*[*x*].
- The successor of the largest key is NIL.
- Search consists of two cases.
  - » If node *x* has a non-empty right subtree, then *x*'s successor is the minimum in the right subtree of *x*.
  - » If node *x* has an empty right subtree, then:
    - As long as we move to the left up the tree (move up through right children), we are visiting smaller keys.
    - *x*'s successor *y* is the node that is the predecessor of *x* (*x* is the maximum in *y*'s left subtree).
    - In other words, *x*'s successor *y*, is the lowest ancestor of *x* whose left child is also an ancestor of *x* or is *x* itself.

Successor

Case 1: *x* has a non-empty right subtree.



Case 2: *x* has an empty right subtree.



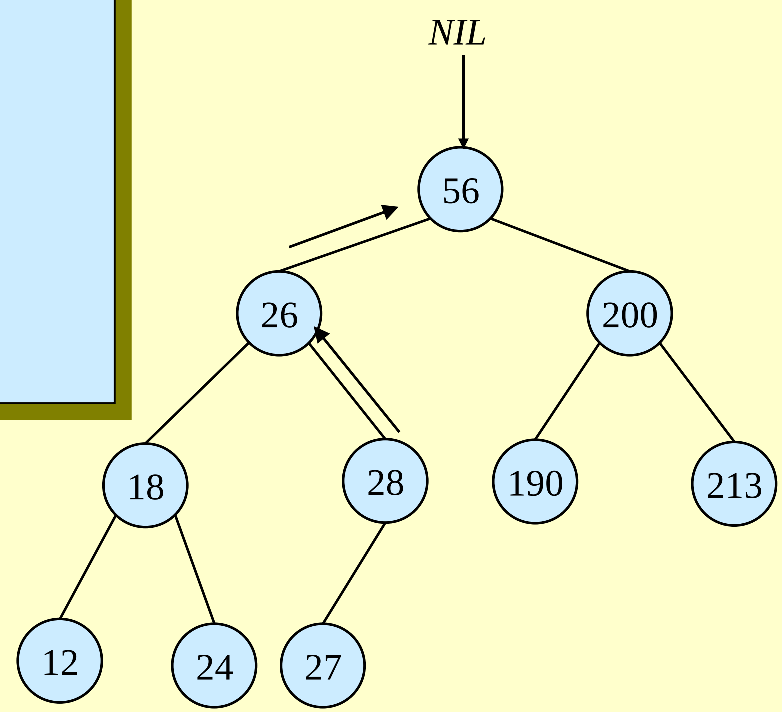
Pseudo-code for Successor

Tree-Successor(x)

```
if right[x] ≠ NIL
    then return Tree-Minimum(right[x])
y ← p[x]
while y ≠ NIL and x = right[y]
do x ← y
    y ← p[y]
return y
```

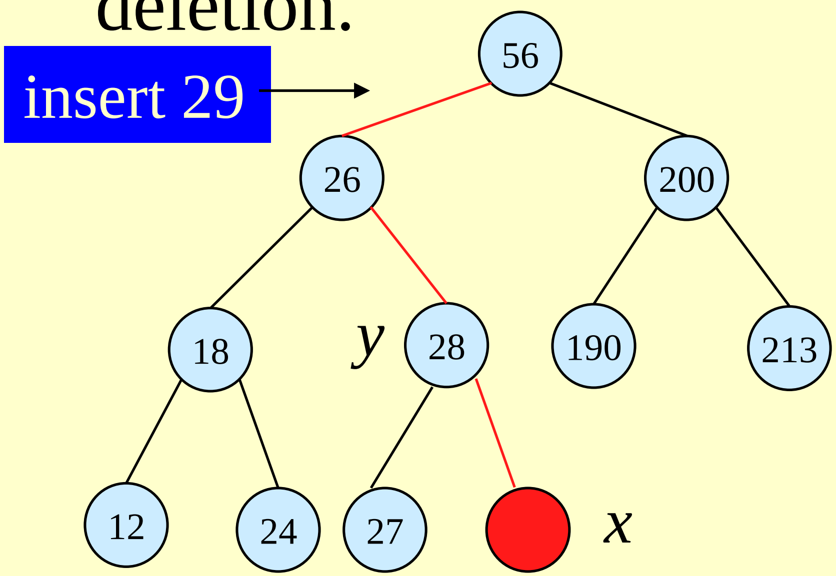
Code for ***predecessor*** is symmetric.

Running time: $O(h)$



BST Insertion – Pseudocode

- Change the dynamic set represented by a BST.
- Ensure the binary-search-tree property holds after change.
- Insertion is easier than deletion.



Tree-Insert(*T*, *z*)

```
y ← NIL
x ← root[T]
while x ≠ NIL
    do y ← x
        if key[z] < key[x]
            then x ← left[x]
            else x ← right[x]
p[z] ← y
if y = NIL
    then root[T] ← z
    else if key[z] < key[y]
        then left[y] ← z
        else right[y] ← z
```

Analysis of Insertion

- Initialization: $O(1)$
- While loop in lines 3-7 searches for place to insert *z*, maintaining parent *y*.
  This takes $O(h)$ time.
- Lines 8-13 insert the value: $O(1)$

⇒ TOTAL: $O(h)$ time to insert a node.

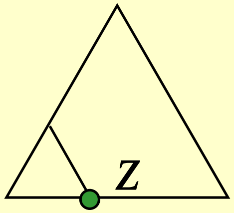
Tree-Insert(*T*, *z*)

```
1.   y ← NIL
2.   x ← root[T]
3.   while x ≠ NIL
4.       do y ← x
5.           if key[z] < key[x]
6.               then x ← left[x]
7.               else x ← right[x]
8.   p[z] ← y
9.   if y = NIL
10.      then root[t] ← z
11.      else if key[z] < key[y]
12.          then  left[y] ← z
13.          else right[y] ← z
```

Tree-Delete (T, z)

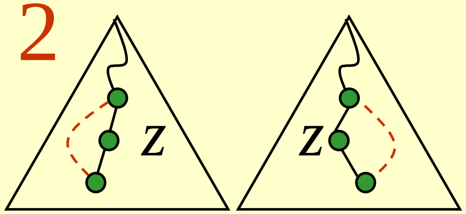
if z has no children ♦ case 1

then remove z



if z has one child ♦ case 2

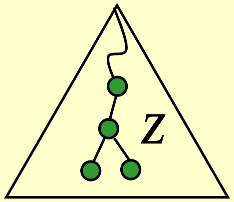
then make $p[z]$ point to child



if z has two children (subtrees) ♦ case 3

then swap z with its successor

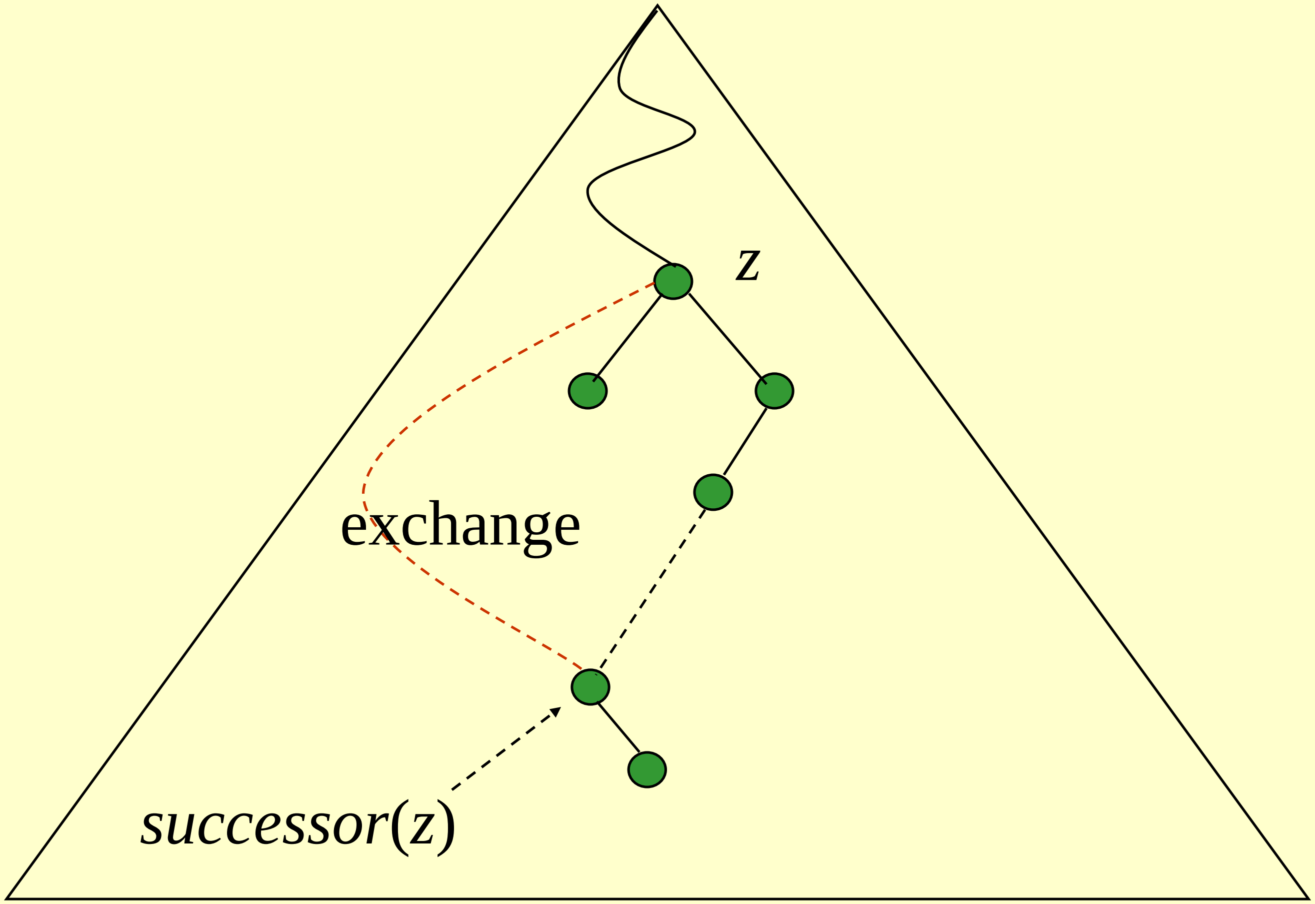
perform case 1 or case 2 to delete it



⇒ TOTAL: $O(h)$ time to delete a node

Tree-Delete (*T*, *z*)

Illustration for case 3:



Deletion – Pseudocode

Tree-Delete(T, z)

```
/* Determine which node to splice out: either z or z's successor. */
if left[z] = NIL or right[z] = NIL
      then y ← z                          /*Case 1 or Case 2*/
      else y ← Tree-Successor[z]      /*Case 3*/
/* Set x to a non-NIL child of y, or to NIL if y has no children. */
if left[y] ≠ NIL                          /*y has one child or no child.*/
      then x ← left[y]                 /*x can be a child of y or NIL.*/
      else x ← right[y]
/* y is removed from the tree by manipulating pointers of p[y]
   and x */
if x ≠ NIL
      then p[x] ← p[y]
/* Continued on next slide */
```

y is the node be deleted, which has at most one child.

x is the unique child of *y*.

Deletion – Pseudocode

Tree-Delete(T, z) (Contd. from previous slide)

```
if p[y] = NIL                          /*if y is the root*/
    then root[T] ← x
    else if  y = left[p[y]]            /*y is a left child.*/
            then left[p[y]] ← x
            else right[p[y]] ← x
/* If z's successor was spliced out, copy its data into z */
if y ≠ z                               /*y is z's successor.*/
    then  key[z] ← key[y]
          copy y's satellite data into z.
return y
```

Correctness of Tree-Delete

- How do we know case 2 should go to case 1 or case 2 instead of back to case 3?
  - » Because when *x* has 2 children, its successor is the minimum in its right subtree, and that successor has no left child (hence 1 or 2 child).
- Equivalently, we could swap with predecessor instead of successor. It might be good to alternate to avoid creating lopsided tree.

More on tree traversal

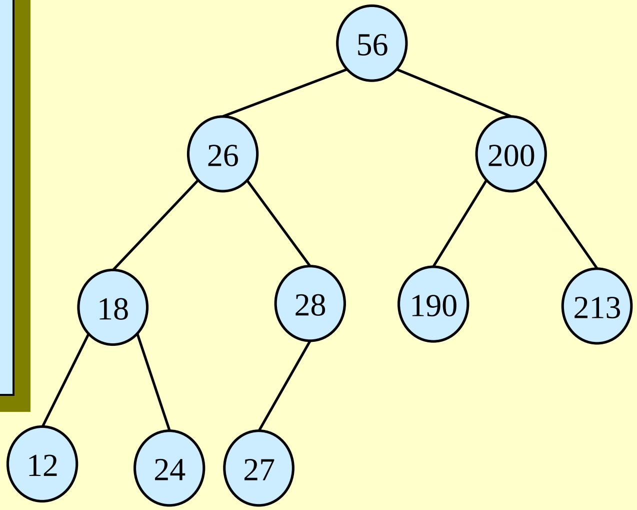
<u>preOrder-Tree-Walk (*x*)</u>

1. **if** $x \neq$ NIL

2. **then** print *key*[*x*]

3. preOrder-Tree-Walk(*left*[*x*])

4. preOrder-Tree-Walk(*right*[*x*])



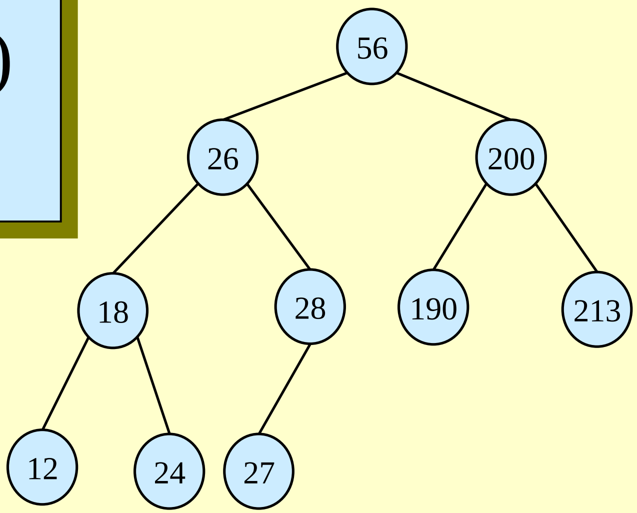
Also called top-down searching, depth-first searching

More on tree traversal

<u>postOrder-Tree-Walk (*x*)</u>

1. **if** $x \neq$ NIL
2. **then** postOrder-Tree-Walk(*left*[*x*])
3. postOrder-Tree-Walk(*right*[*x*])
4. print *key*[*x*]

Also called bottom-up searching

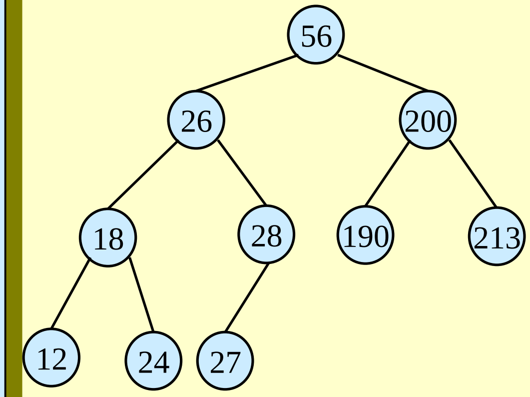


More on tree traversal

<u>Breadth-first (x)</u>

enqueue(Q, x)
while $Q \neq$ empty **do**
    v := **dequeue**(Q)
    print *key*[x]
    Let $v_1, \ldots, v_k$ be the children of v
    for ($i = 1$ to k) **enqueue**(Q, v_i)

Q is a queue.



More on tree traversal

<u>Depth-first(x)</u> (recursive)

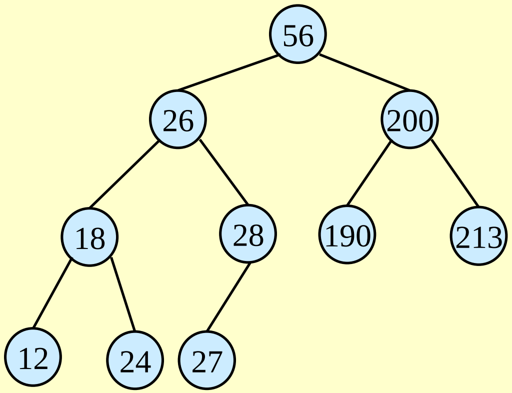
Algorithm DFS(x)

1. **if** $x \neq$ NIL

2. **then** print $key[x]$

3. Let $v_1, \ldots, v_k$ be the children of x

4. **for** ($i = k$ to 1) DFS(v_i)



More on tree traversal

<u>Depth-first(x)</u>   (non-recursive)

1. **push**(S, x)

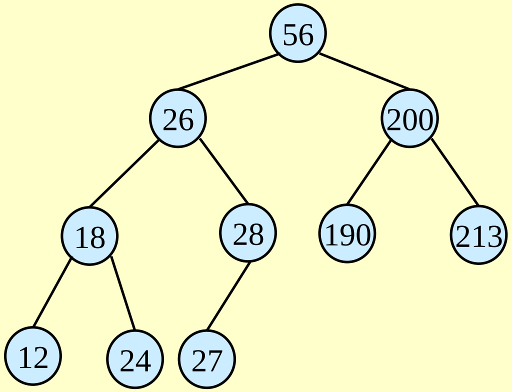
2. **while** $S \neq$ empty **do**

3. v := **pop**(S)

4.   print $key[x]$

5.   Let $v_1, \ldots, v_k$ be the children of x

6.   **for** ($i = k$ to 1) **push**(S, v_i)



S is a stack.

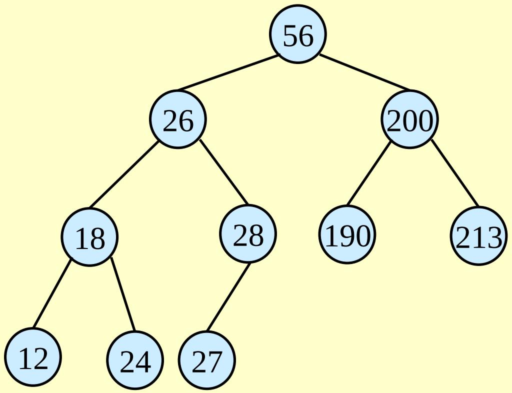
It is also called the preoreder search and top-down search.

More on tree traversal

<u>PostOrder(x)</u>   (recursive)

Algorithm PostOrder(x)

1. **if** $x \neq$ NIL
2. **then** Let $v_1, \ldots, v_k$ be the children of x
3. **for** ($i = k$ to 1) PostOrder(v_i)
4. Print $key[x]$

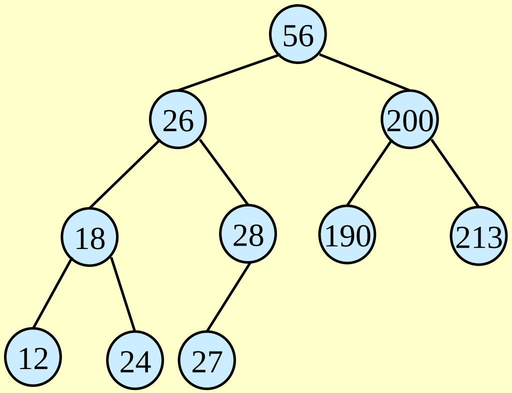


It is also called the bottom-up search.

More on tree traversal

PostOrder(x)   (non-recursive)

1. **push**(S, x)
2. **while** $S \neq$ empty **do**
3. v := **top**(S)
4.   if v is leaf or marked
5.   then print *key*[v], **pop**(S)
6.   else *mark* v
7.   Let $v_1, \ldots, v_k$ be the children of v
8.   **for** ($i = k$ to 1) **push**(S, v_i)



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Binary Search Trees

- View today as data structures that can support dynamic set operations.
  - » Search, Minimum, Maximum, Predecessor, Successor, Insert, and Delete.
- Can be used to build
  - » Dictionaries.
  - » Priority Queues.
- Basic operations take time proportional to the height of the tree – $O(h)$.

Red-black trees: Overview

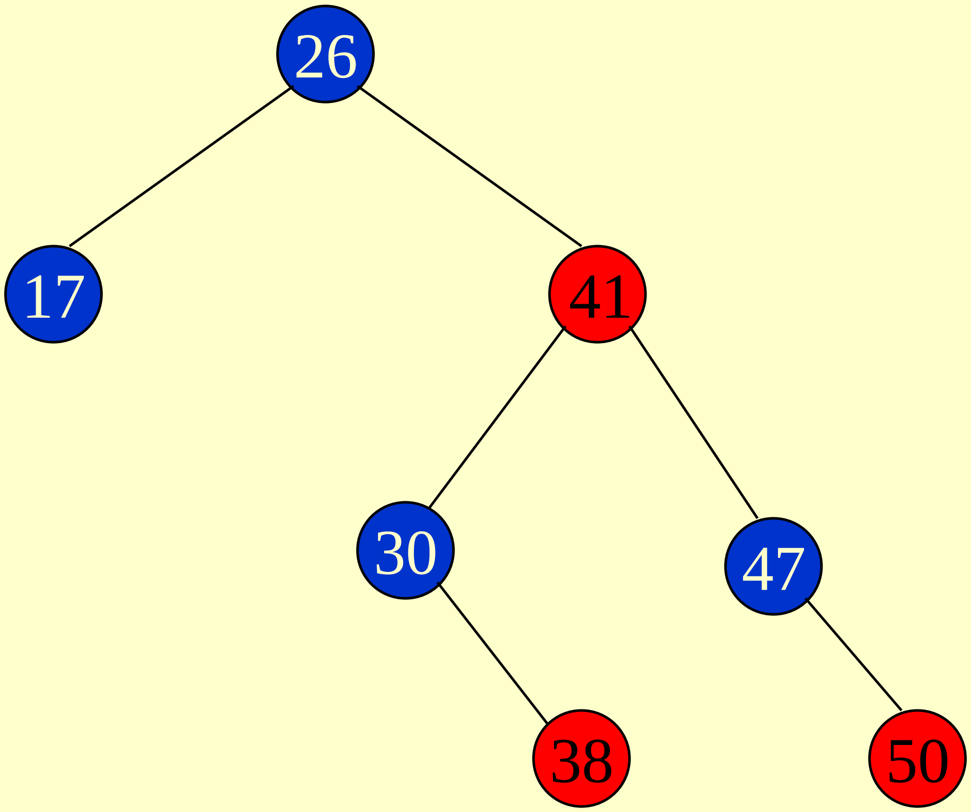
- Red-black trees are a variation of binary search trees to ensure that the tree is ***balanced***.
  - » Height is $O(\lg n)$, where n is the number of nodes.
- Operations take $O(\lg n)$ time in the worst case.

Red-black Tree

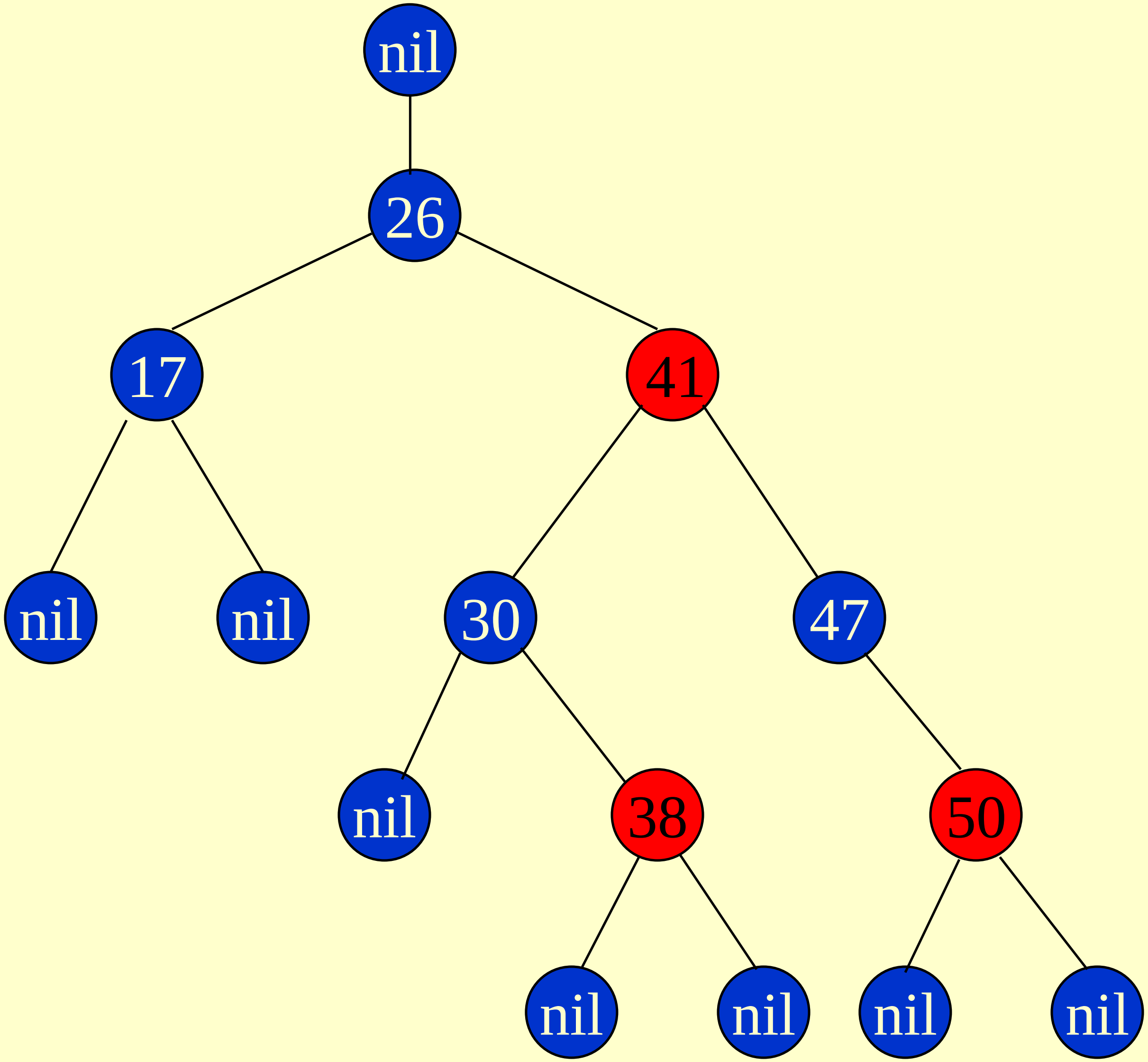
- Binary search tree + 1 bit per node: the attribute *color*, which is either **red** or **black**.
- All other attributes of BSTs are inherited:
  - » *key*, *left*, *right*, and *p*.

- If a child or the parent of a node does not exist, the corresponding pointer field of the node contains the value *nil*.
- Sentinel - *nil*[*T*], representing all the *nil* nodes.

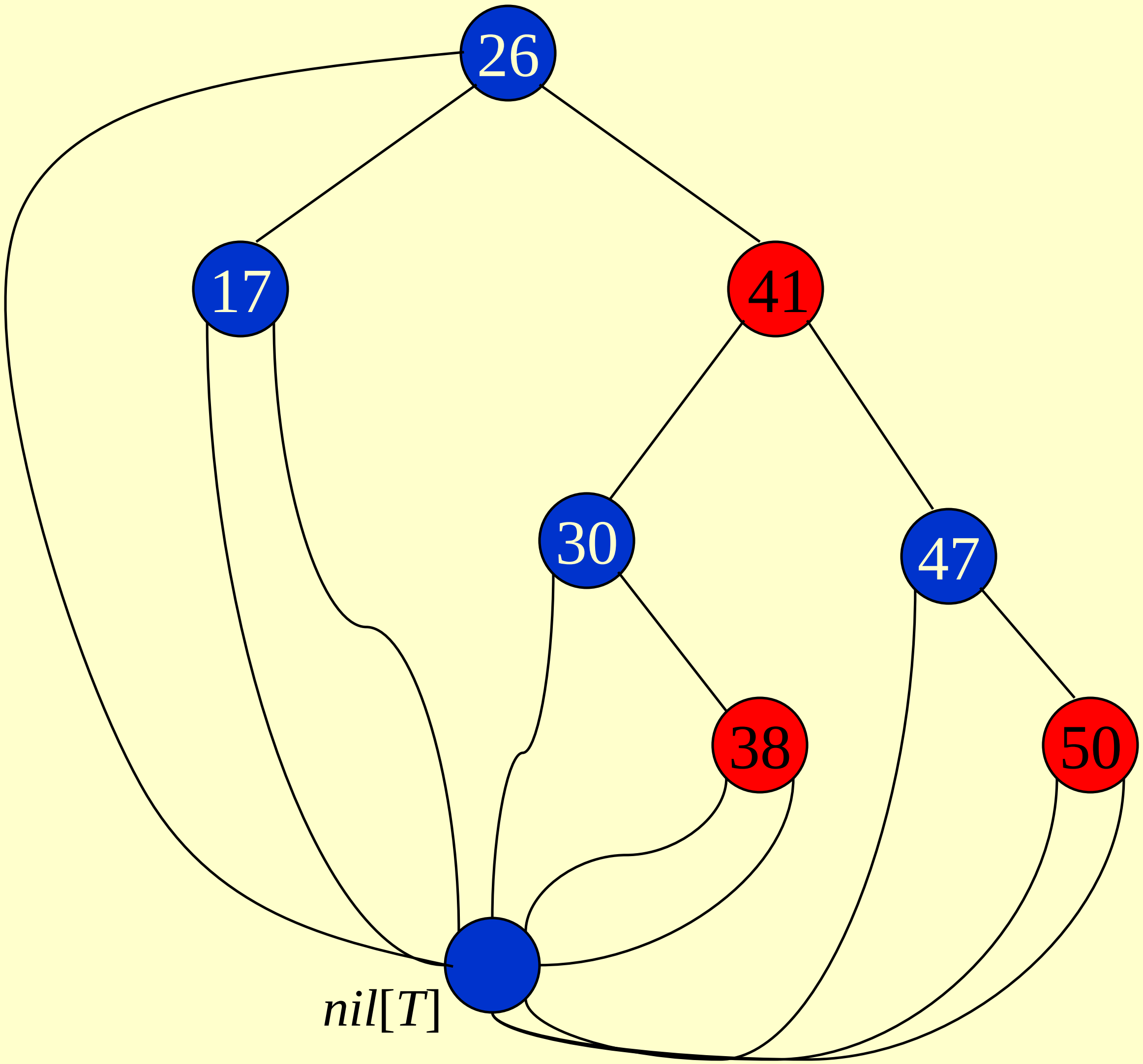
Red-black Tree – Example



Red-black Tree – Example



Red-black Tree – Example



Red-black Properties

1. Every node is either red or black.
2. The root is black.
3. Every leaf (*nil*) is black.
4. If a node is red, then both its children are black.

5. For each node, all paths from the node to descendant leaves contain the same number of black nodes.

Height of a Red-black Tree

- Height of a node:
  - Number of edges in a longest path to a leaf.
- Black-height of a node x, $bh(x)$:
  - $bh(x)$ is the number of black nodes (including $nil[T]$) on the path from x to leaf, not counting x.
- Black-height of a red-black tree is the black-height of its root.
  - By Property 5, black height is well defined.

Height of a Red-black Tree

- Example:
- Height of a node:
  - » Number of edges in a longest path to a leaf.
- Black-height of a node *bh*(*x*) is the number of black nodes on path from *x* to leaf, not counting *x*.