Recursive Equation

- Recurrence relations
- How to solve a recursive equation

Analysis of Merge Sort

- Running time ***T*(*n*)** of Merge Sort:
- Divide: computing the middle takes $\Theta(1)$
- Conquer: solving 2 subproblems takes $2T(n/2)$
- Combine: merging *n* elements takes $\Theta(n)$
- Total:

$$T(n) = \Theta(1) \qquad \text{if } n = 1$$
$$T(n) = 2T(n/2) + \Theta(n) \qquad \text{if } n > 1$$

$\Rightarrow T(n) = \Theta(n \lg n)$ (CLRS, Chapter 4)

Recurrence Relations

- Equation or an inequality that characterizes a function by its values on smaller inputs.
- **Solution Methods** (Chapter 4)
  - Substitution Method.
  - Recursion-tree Method.
  - Master Theorem Method.
- Recurrence relations **arise when we analyze the running time of iterative or recursive algorithms**.
  - **Ex:** Divide and Conquer.

$$T(n) = \Theta(1) \quad \text{if } n \le c$$
$$T(n) = a\,T(n/b) + D(n) \quad \text{otherwise}$$

Substitution Method

- **Guess** the form of the solution, then **use mathematical induction** to show it correct.
  - Substitute guessed answer for the function when the inductive hypothesis is applied to smaller values.
- Works well when the solution is easy to guess.
- No general way to guess the correct solution.

Example – Exact Function

Recurrence: $T(n) = 1$ if $n = 1$

$T(n) = 2T(n/2) + n$ if $n > 1$

- Guess: $T(n) = n \lg n + n$.
- Induction:
  - **Basis:** $n = 1 \Rightarrow n \lg n + n = 1 = T(n)$.
  - **Hypothesis:** $T(k) = k \lg k + k$ for all $k < n$.
  - **Inductive Step:**

$$
\begin{aligned}
T(n) &= 2\,T(n/2) + n \\
&= 2\,((n/2)\lg(n/2) + (n/2)) + n \\
&= n\,(\lg(n/2)) + 2n \\
&= n \lg n - n + 2n \\
&= n \lg n + n
\end{aligned}
$$

Recursion-tree Method

- Making a good guess is sometimes difficult with the substitution method.
- Use **recursion trees** to devise good guesses.
- Recursion Trees
  - Show successive expansions of recurrences using trees.
  - Keep track of the time spent on the subproblems of a divide and conquer algorithm.
  - Help organize the algebraic bookkeeping necessary to solve a recurrence.

Recursion Tree – Example

- Running time of Merge Sort:

$$T(n) = \Theta(1) \qquad \text{if } n = 1$$

$$T(n) = 2T(n/2) + \Theta(n) \qquad \text{if } n > 1$$

- Rewrite the recurrence as

$$T(n) = \boldsymbol{c} \qquad \text{if } n = 1$$

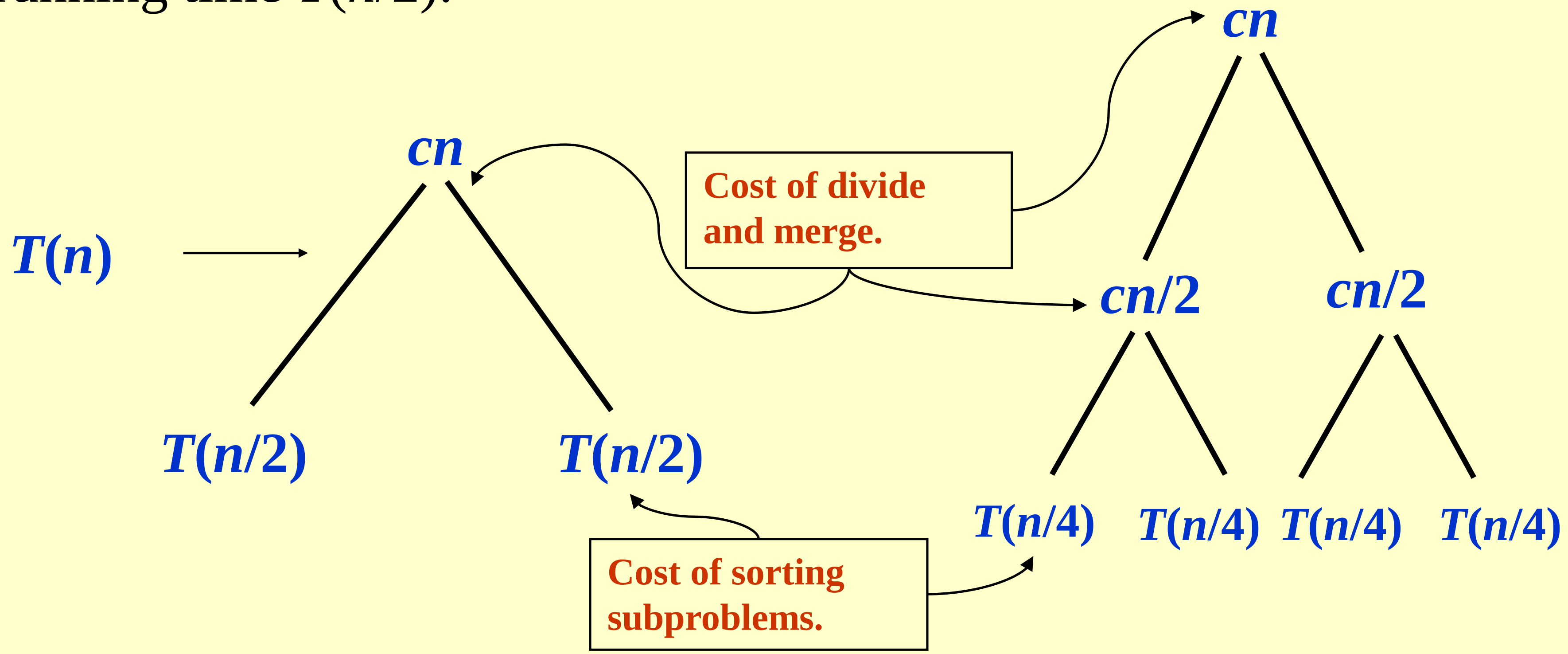
$$T(n) = 2T(n/2) + \boldsymbol{cn} \qquad \text{if } n > 1$$

$\boldsymbol{c > 0}$: Running time for the base case and time per array element for the divide and combine steps.

Recursion Tree for Merge Sort

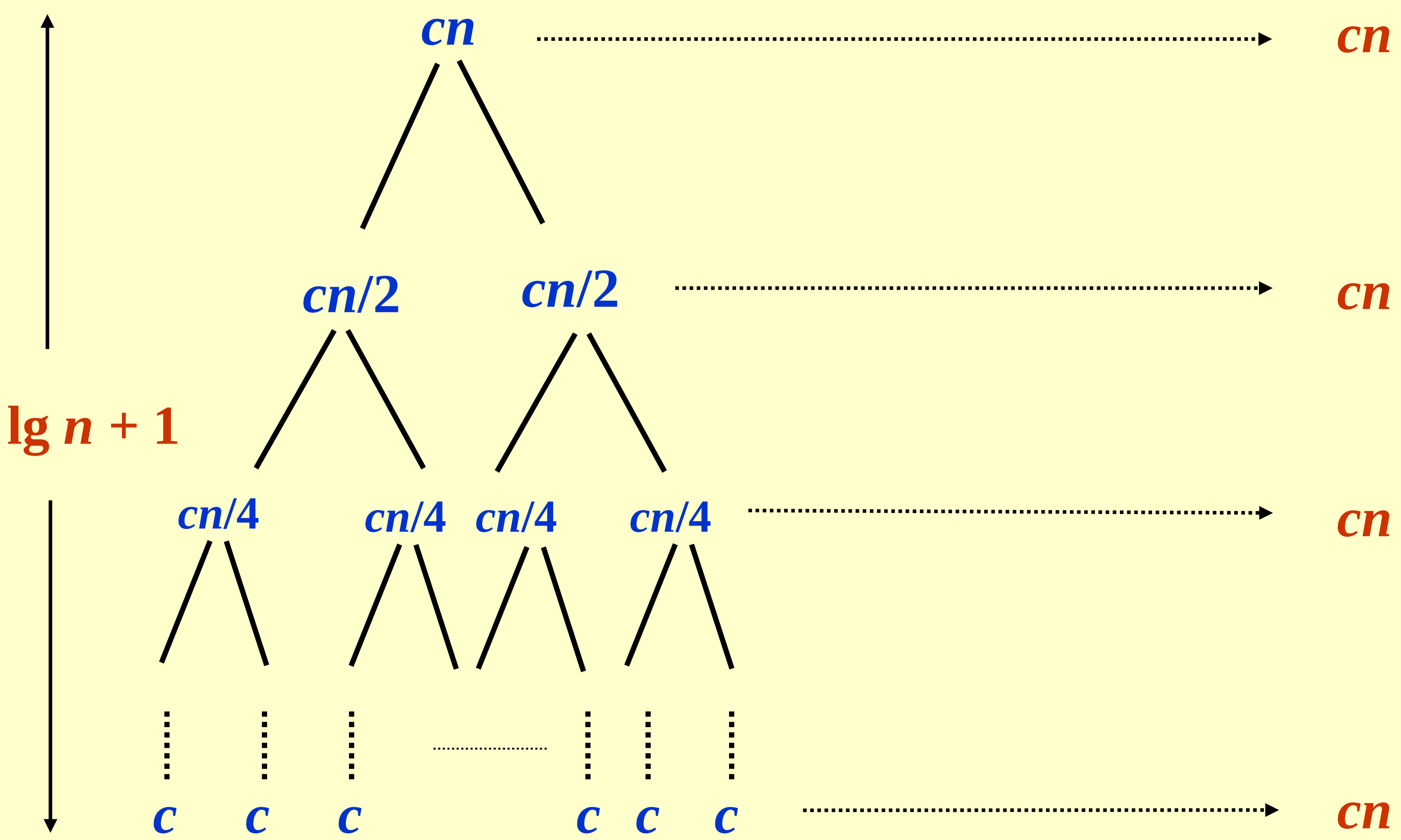
For the original problem, we have a cost of cn, plus two subproblems each of size $(n/2)$ and running time $T(n/2)$.

Each of the size $n/2$ problems has a cost of $cn/2$ plus two subproblems, each costing $T(n/4)$.



Recursion Tree for Merge Sort

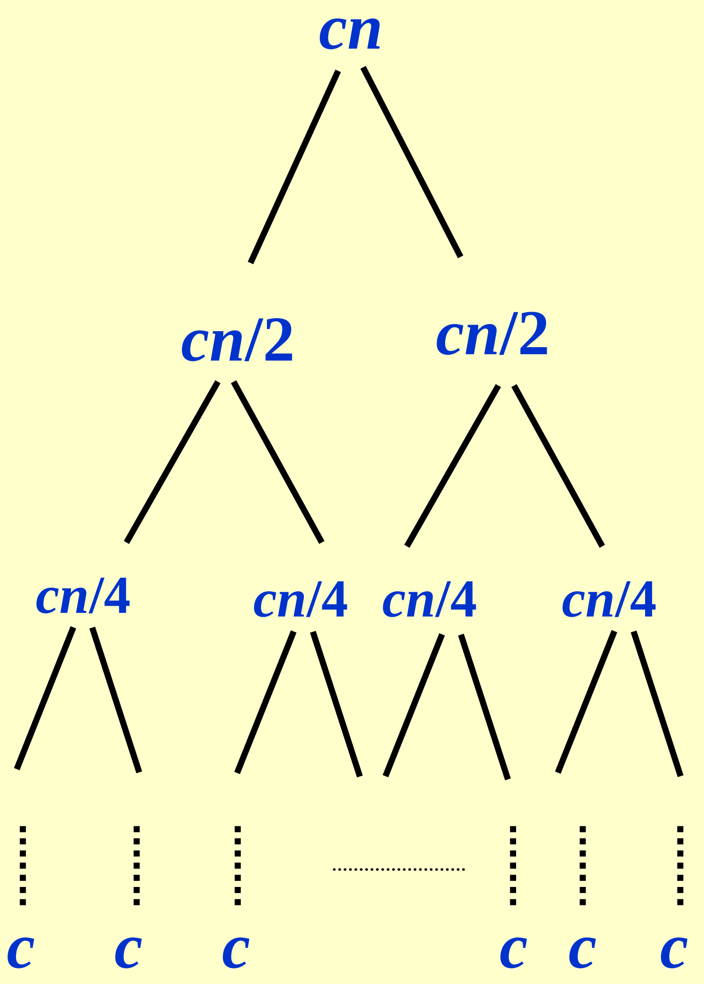
Continue expanding until the problem size reduces to 1.



Total: $cn\lg n+cn$

Recursion Tree for Merge Sort

Continue expanding until the problem size reduces to 1.



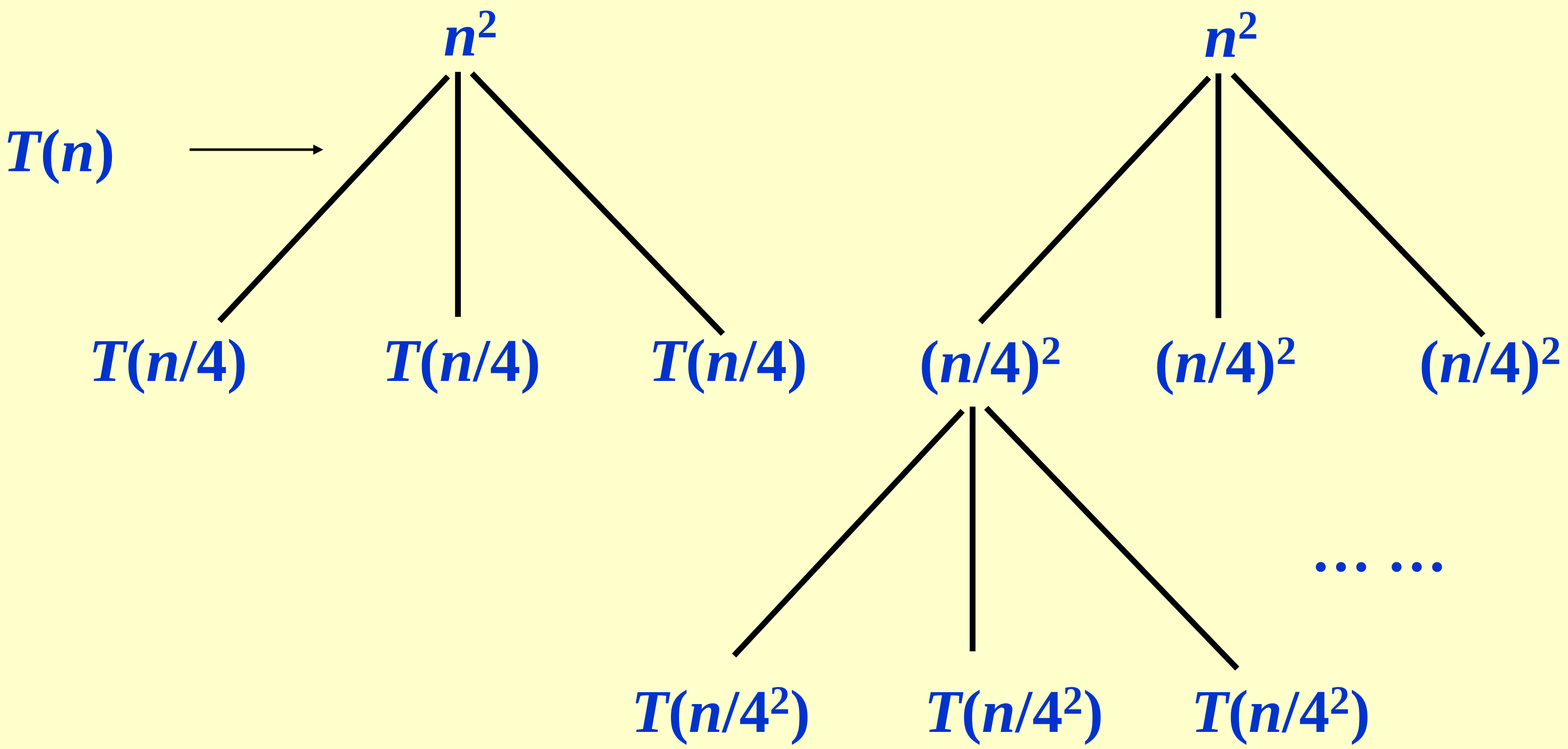
- Each level has total cost ***cn***.
- Each time we go down one level, the number of subproblems doubles, but the cost per subproblem halves $\Rightarrow$ *cost per level remains the same*.
- There are lg n + 1 levels, height is lg n. (Assuming n is a power of 2.)
- Can be proved by induction.
- Total cost = sum of costs at each level = $(\lg n + 1)cn = cn\lg n + cn = \Theta(n \lg n)$.

Other Examples

- Use the recursion-tree method to determine a guess for the recurrences
  - $T(n) = 3T(\lfloor n/4 \rfloor) + \Theta(n^2)$.
  - $T(n) = T(n/3) + T(2n/3) + O(n)$.

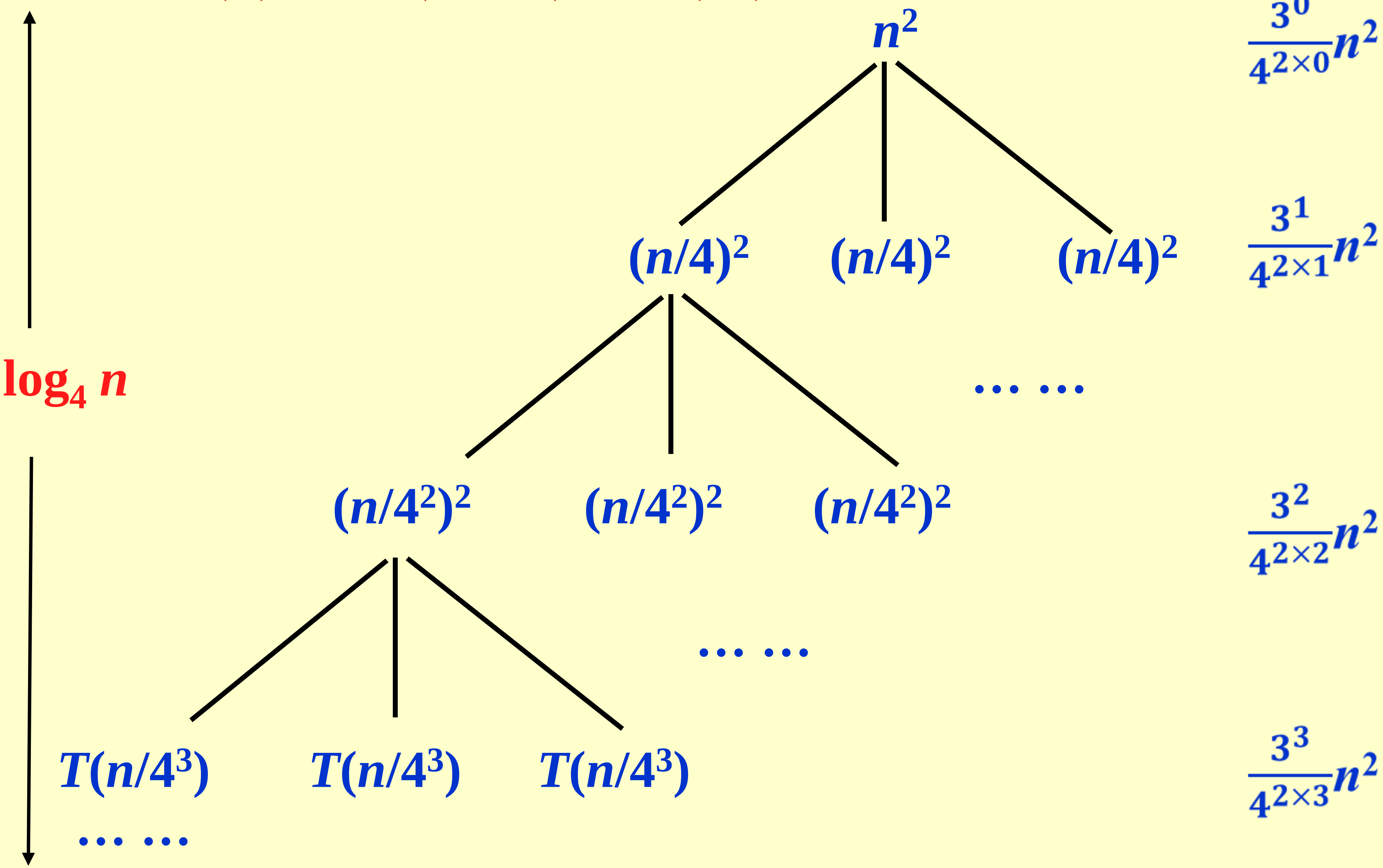
Other Examples

- $T(n) = 3T(\lfloor n/4 \rfloor) + \Theta(n^2)$.



Other Examples

- $T(n) = 3T(\lfloor n/4 \rfloor) + \Theta(n^2)$.



Other Examples

- $T(n) = 3T(\lfloor n/4 \rfloor) + \Theta(n^2).$

$T(n) = 3T(n/4) + n^2$

$= n^2 + \frac{3^1}{4^{2\times 1}}n^2 + 3T(n/4^2)$

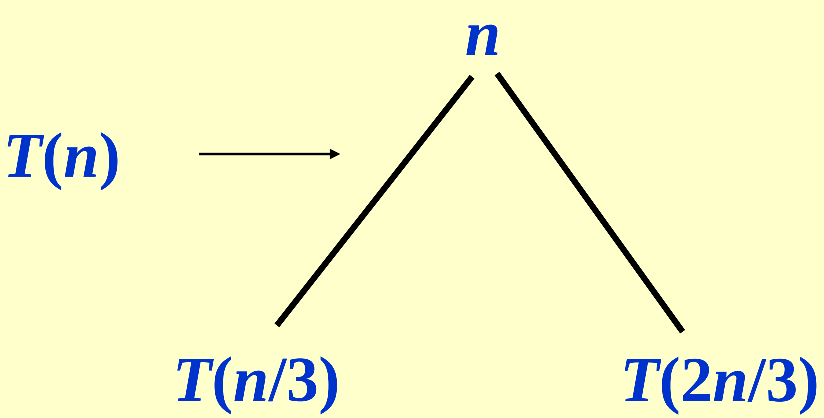
$= n^2 + \frac{3^1}{4^{2\times 1}}n^2 + \frac{3^2}{4^{2\times 2}}n^2 + \ldots + \frac{3^i}{4^{2\times i}}n^2 + \ldots + \frac{3^{log_4 n}}{4^{2\times log_4 n}}\,n^2$

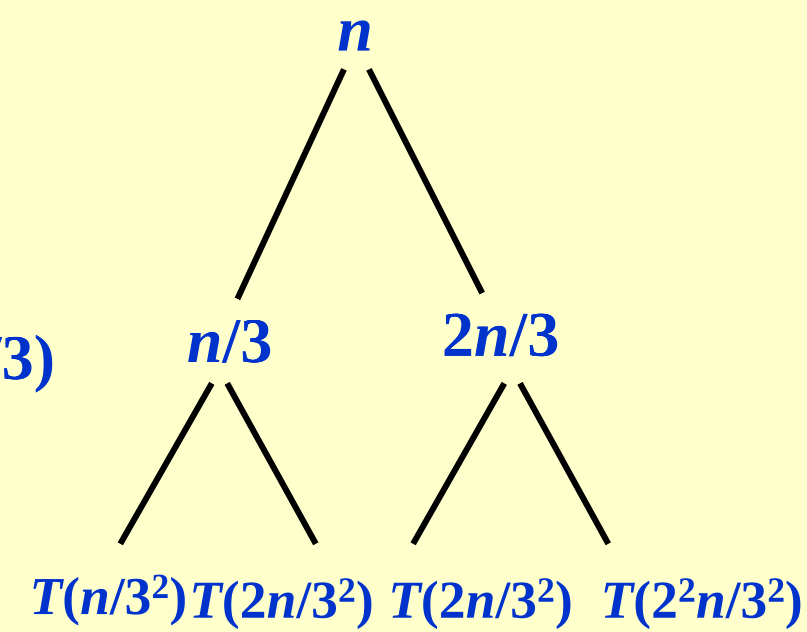
$= (\frac{n}{4})^2(1 + \frac{3}{4} + (\frac{3}{4})^2 + + \ldots + (\frac{3}{4})^{log_4 n})$

$= (\frac{n}{4})^2\frac{(1-\frac{3}{4})^{log_4 n}}{1-\frac{3}{4}} = \Theta(n^2).$

Other Examples

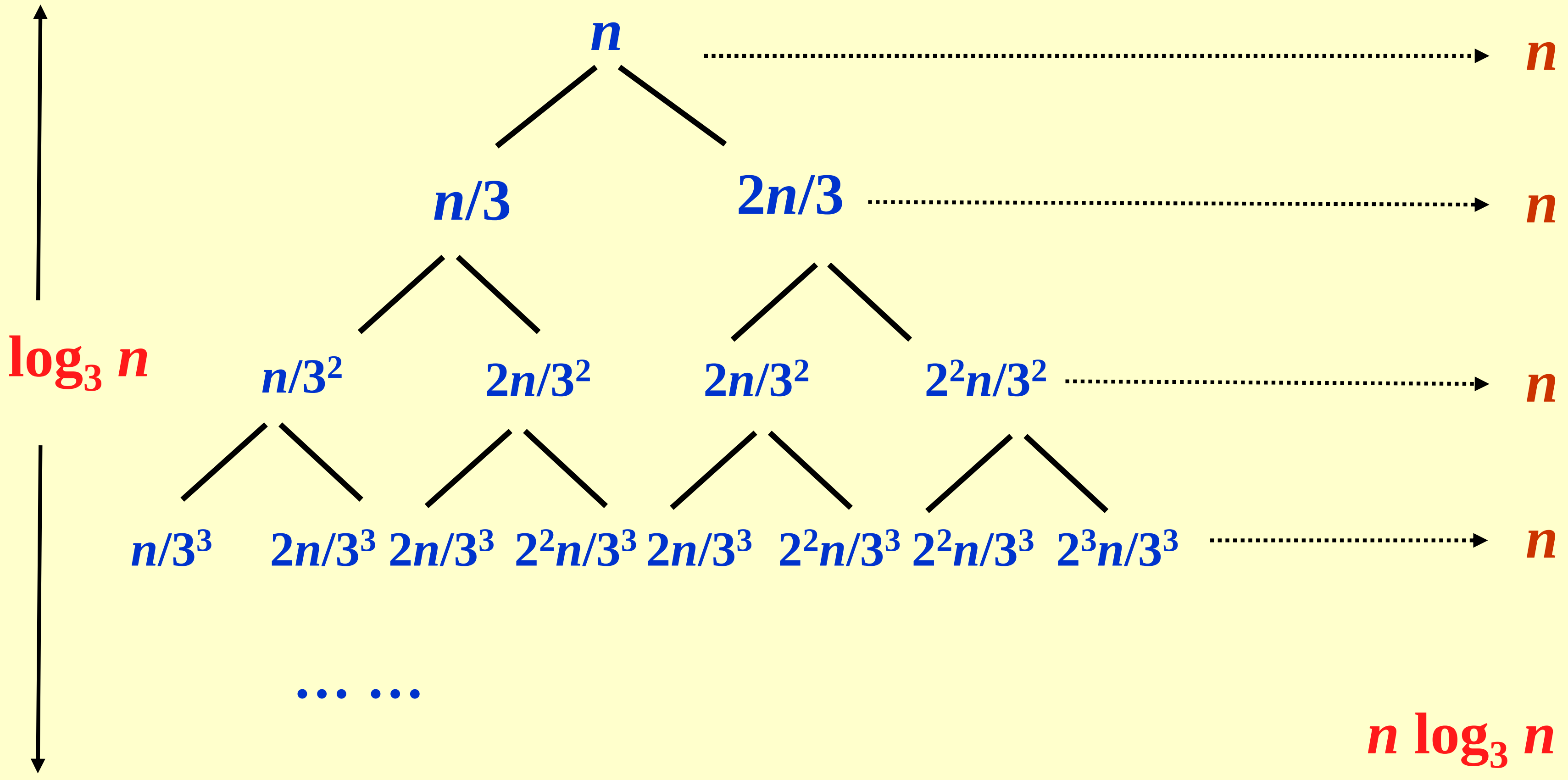
- $T(n) = T(n/3) + T(2n/3) + O(n)$.





Other Examples

- $T(n) = T(n/3) + T(2n/3) + O(n)$.



Other Examples

- $T(n) = T(n/3) + T(2n/3) + O(n)$

$T(n) = T(n/3) + T(2n/3) + n$

$T(n) = (T(n/3^2) + T(2n/3^2) + n/3) + (T(2n/3^2) + T(2^2n/3^2) + 2n/3) + n$

$= T(n/9) + 2T(2n/9) + T(4n/9) + 2$

$= (T(n/27) + T(2n/27) + n/9) + 2(T(2n/27) + T(4n/27) + 2n/9) + (T(4n/27) + T(8n/27) + 4n/9) + 2n$

$= T(n/27) + 3T(2n/27) + 3T(4n/27) + T(8n/27) + 3n$

$= \ldots \ldots + n/27 + 3(2n/27) + 3(4n/27) + 8n/27 + 3n$

$= n \log_3 n$

Recursion Trees – Caution Note

- Recursion trees only generate guesses.
  - Verify guesses using substitution method.
- A small amount of "sloppiness" can be tolerated. Why?
- If careful when drawing out a recursion tree and summing the costs, it can be used as direct proof.

The Master Method

- Based on the Master theorem.
- "Cookbook" approach for solving recurrences of the form

  $T(n) = aT(n/b) + f(n)$

  - $a \geq 1$, $b > 1$ are constants.
  - $f(n)$ is asymptotically positive.
  - n/b may not be an integer, but we ignore floors and ceilings. Why?
- Requires memorization of three cases.

The Master Theorem

Theorem 4.1

Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined on nonnegative integers by the recurrence $T(n) = aT(n/b) + f(n)$, where we can replace n/b by $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. $T(n)$ can be bounded asymptotically in three cases:

1. If $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.
2. If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \lg n)$.
3. If $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$, and if, for some constant $c < 1$ and all sufficiently large n, we have $a \cdot f(n/b) \leq c\, f(n)$, then $T(n) = \Theta(f(n))$.

We'll return to recurrences as we need them…

The Master Method

$$T(n) = aT(n/b) + f(n)$$

$$= a(T(n/b^2) + f(n/b)) + f(n)$$

$$= a(a(T(n/b^3) + f(n/b^2)) + f(n/b)) + f(n)$$

$$\dots \quad \dots$$

$$\leq c^{\log_b a} f(n) + c^{\log_b a - 1} f(n) + \dots + c^2 f(n) + cf(n) + f(n)$$

$$= f(n)\,(c^{\log_b a} + c^{\log_b a - 1} \dots + c)$$

$$= \Theta(f(n))$$

$$a \cdot f(n/b) \leq c\, f(n) \quad (c < 1),$$

$$a^m f(n/b^{m+1}) \leq c a^{m-1} f(n/b^m) \leq \dots \leq c^{m+1} f(n).$$