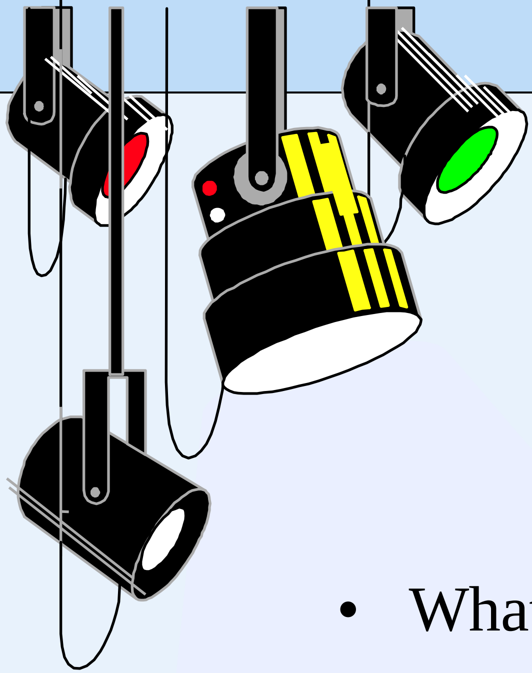


Outline: Reachability Query Evaluation

- What is reachability query?
- Reachability query evaluation based on matrix multiplication
- Strassen's algorithm (for matrix multiplication)
- Warren's algorithm (for generating transitive closures)
- Reachability based on tree encoding

Motivation

- **Efficient method to evaluate graph reachability queries**

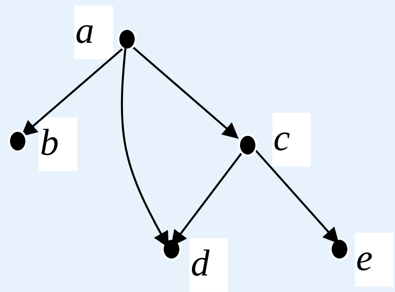
  Given a directed graph G, check whether a node v is reachable from another node u through a path in G.

- **Application**
  - **XML data processing**
  - **Type checking in object-oriented languages and databases**
  - **Geographical data navigation**
  - **Internet routing**
  - **Social network**

A simple method

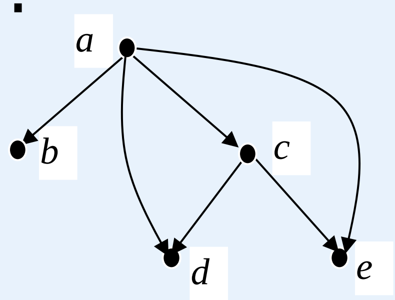
- *store a transitive closure as a matrix*

G:



$$M = \begin{array}{c|ccccc} & a & b & c & d & e \\ \hline a & 0 & 1 & 1 & 1 & 0 \\ b & 0 & 0 & 0 & 0 & 0 \\ c & 0 & 0 & 0 & 1 & 1 \\ d & 0 & 0 & 0 & 0 & 0 \\ e & 0 & 0 & 0 & 0 & 0 \end{array}$$

G^*:



The transitive closure G^* of a graph G is a graph such that there is an edge (u, v) in G^* iff there is path from u to v in G.

$$M^* = \begin{array}{c|ccccc} & a & b & c & d & e \\ \hline a & 0 & 1 & 1 & 1 & 1 \\ b & 0 & 0 & 0 & 0 & 0 \\ c & 0 & 0 & 0 & 1 & 1 \\ d & 0 & 0 & 0 & 0 & 0 \\ e & 0 & 0 & 0 & 0 & 0 \end{array}$$

Matrix Multiplication

- **Definition**

  - Two matrices A and B are compatible if the number of columns of A equals the number of B.
  - If $A = (a_{ij})$ is an $m \times n$ matrix and $B = (b_{ij})$ is an $n \times p$ matrix, then their matrix product $C = A \times B$ is an $m \times p$ matrix $C = (c_{ik})$ such that

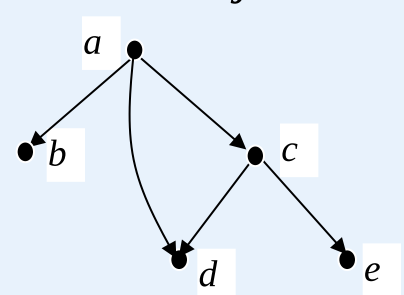
$$c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$$

for $i = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, p$.

$$M \times M = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Each entry (i, j) in $M \times M$ represents a path of length 2 from i to j.

G:



Each entry (i, j) in $M \times M$ represents a path of length 2 from i to j.

Each entry (i, j) in $M \times M \times M$ represents a path of length 3 from i to j.

$\vdots$

Each entry (i, j) in $\overbrace{M \times M \times M \ldots \times M}^{k}$ represents a path of length k from i to j.

Define:

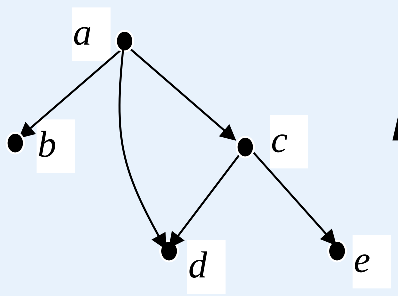
$$\boldsymbol{M^*} = M^{(1)} \vee M^{(2)} \vee M^{(3)} \vee \ldots \vee M^{(n)}$$

Each entry (i, j) in $\boldsymbol{M^*}$ represents a path from i to j.

Time overhead: $O(n^4)$.
Space overhead: $O(n^2)$.
Query time: $O(1)$.

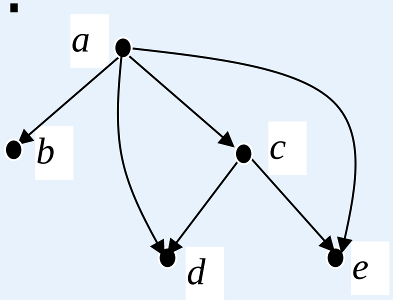
Example

G:



$$M = \begin{array}{cc} & \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} & \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

G^*:



$$M^* = M \vee (M \times M) = \begin{array}{cc} & \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} & \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Each entry (i, j) in P represents a path from i to j.

Strassen's Algorithm

Strassen's algorithm runs in $O(n^{\lg 7}) = O(n^{2.81})$ time. For sufficiently large values of n, it outperforms Warren's algorithm.

- **An overview of the algorithm**

Strassen's algorithm can be viewed as an application of a familiar design technique: divide and conquer. Consider the computation $C = A \times B$, where A, B, and C are $n \times n$ matrices. Assuming that n is an exact power of 2, we divide each of A, B, and C into four $n/2 \times n/2$ matrices, rewriting the equation $C = A \times B$ as follows:

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

$$r = ae + bg$$
$$s = af + bh$$
$$t = ce + dg$$
$$u = af + dh$$

Each of these four equations specifies two multiplications of $n/2 \times n/2$ matrices and the addition of their $n/2 \times n/2$ products.
So the time complexity of the algorithm satisfies the following recursive equation:

$$T(n) = 8T(n/2) + O(n^2)$$

The solution of this equation is $T(n) = O(n^3)$.

Strassen discovered a different approach that requires only 7 recursive multiplications of $n/2 \times n/2$ matrices and $O(n^2)$ scalar additions and subtractions, yielding the recurrence:

$$\begin{aligned} T(n) &= 7T(n/2) + O(n^2) \\ &= O(n^{\lg 7}) \\ &= O(n^{2.81}). \end{aligned}$$

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

Strassen's algorithm works in four steps:

1. Divide the input matrices A and B into $n/2 \times n/2$ matrices.

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{pmatrix} \qquad \boldsymbol{B} = \begin{pmatrix} \boldsymbol{e} & \boldsymbol{f} \\ \boldsymbol{g} & \boldsymbol{h} \end{pmatrix}$$

2. Using O(n^2) scalar additions and subtractions, compute 14 matrices A_1, B_1, A_2, B_2, …, A_7, B_7, each of which is $n/2 \times n/2$.

$$\begin{array}{ll}
A_1 = a, & B_1 = (f - h), \\
A_2 = (a + b), & B_2 = h, \\
A_3 = (c + d), & B_3 = e, \\
A_4 = d, & B_4 = (g - d), \\
A_5 = (a + d), & B_5 = (e + h), \\
A_6 = (b - d), & B_6 = (g + h), \\
A_7 = (c - a) & B_7 = (e + f)
\end{array}$$

Strassen's algorithm works in four steps:

3. Recursively compute the seven matrix products

$$P_i = A_i \times B_i \text{ for } i = 1, 2, \ldots, 7.$$

4. Compute the desired submatrices ***r***, ***s***, ***t***, ***u*** of the result matrix ***C*** by adding and/or subtracting various combinations of the P_i matrices, using only $O(n^2)$ scalar additions and subtraction.

$$\boldsymbol{r} = ae + bg = P_5 + P_4 - P_2 + P_6,$$
$$\boldsymbol{s} = af + bh = P_1 + P_2,$$
$$\boldsymbol{t} = ce + dg = P_3 + P_4,$$
$$\boldsymbol{u} = af + dh = P_5 + P_1 - P_3 + P_7.$$

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

Altogether 7 matrix multiplication,
18 matrix additions and subtractions.

$T(n) = 7T(n/2) + O(n^2)$

Assume that $n = 2^m$. We have

$$T(2^m) = 7T(2^{m-1}) + 18(2^{m-1})^2.$$

$$A_m = 7A_{m-1} + 18(2^{m-1})^2, \qquad A_1 = 18.$$

$$G(x) = A_1 + A_2x + A_3x^2 + \ldots$$

Generating function

$$= A_1 + (7A_1 + 18\cdot 2^2)x$$

$$+ (7A_2 + 18\cdot 2^3)x^2$$

$$\ldots \ldots$$

$$= 8 + 7x\,(A_1 + A_2x + A_3x^2 + \ldots) + 18\cdot 4x/(1 - 4x)$$

$$= 8 + 7x\,G(x) + 18\cdot 4x/(1 - 4x)$$

$$(1 - 7x)G(x) = 18(4x/(1 - 4x) + 1) = 18/(1 - 4x)$$

Reachability Queries

$$(1 - 7x)G(x) = 18(4x/(1 – 4x) + 1) = 18/(1 – 4x)$$

$$G(x) = 18/(1 – 4x)(1 – 7x) = 18\left(\frac{-4/3}{1 – 4x} + \frac{7/3}{1 – 7x}\right)$$

$$G(x) = 6\sum_{k=0}^{\infty}(7^{k+1} – 4^{k+1})x^k = A_1 + A_2x + A_3x^2 + \ldots$$

$$\Downarrow$$

$$A_m = 6(7^m – 4^m), \qquad m = \log_2 n$$

$$= O(6\cdot 7^{\log_2 n})$$

$$= O(6\cdot n^{\log_2 7})$$

$$= O(n^{2.81})$$

- **Determining the submatrix products**

It is not clear exactly how Strassen discovered the submatrix products that are the key to making his algorithm work. Here, we reconstruct one plausible discovery method.

Write $P_i = A_i \times B_i$

$$= (\alpha_{i1}a + \alpha_{i2}b + \alpha_{i3}c + \alpha_{i4}d)\ (\beta_{i2}e + \beta_{i1}f + \beta_{i3}g + \beta_{i4}h),$$

where the coefficients α_{ij}, β_{ij} are all drawn from the set {-1, 0, 1}. We guess that each product is computed by adding or subtracting some of the submatrices of A, adding or subtracting some of submatrices of B, and then multiplying the two results together.

Reachability Queries

$$P_i = A_i \times B_i = (\alpha_{i1}a + \alpha_{i2}b + \alpha_{i3}c + \alpha_{i4}d)\ (\beta_{i1}e + \beta_{i2}f + \beta_{i3}g + \beta_{i4}h)$$

$$= (a\ b\ c\ d) \begin{pmatrix} \alpha_{i1} \\ \alpha_{i2} \\ \alpha_{i3} \\ \alpha_{i4} \end{pmatrix} (\beta_{i1}\ \beta_{i2}\ \beta_{i3}\ \beta_{i4}) \begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix}$$

$$= (a\ b\ c\ d) \begin{pmatrix} \alpha_{i1}\beta_{i1} & \alpha_{i1}\beta_{i2} & \alpha_{i1}\beta_{i3} & \alpha_{i1}\beta_{i4} \\ \alpha_{i2}\beta_{i1} & \alpha_{i2}\beta_{i2} & \alpha_{i2}\beta_{i3} & \alpha_{i2}\beta_{i4} \\ \alpha_{i3}\beta_{i1} & \alpha_{i3}\beta_{i2} & \alpha_{i3}\beta_{i3} & \alpha_{i3}\beta_{i4} \\ \alpha_{i4}\beta_{i1} & \alpha_{i4}\beta_{i2} & \alpha_{i4}\beta_{i3} & \alpha_{i4}\beta_{i4} \end{pmatrix} \begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix}$$

Reachability Queries

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

$$r = ae + bg$$
$$s = af + bh$$
$$t = ce + dg$$
$$u = af + dh$$

$r = ae + bg$

$$= (a\ b\ c\ d)\begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix}$$

So r is represented by a matrix:

$$\begin{pmatrix} + & \cdot & \cdot & \cdot \\ \cdot & \cdot & + & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

'·' – represents 0.

'+' – represents +1.

'-' – represents -1.

$s = af + bh$ $\qquad$ $t = ce + dg$ $\qquad$ $s = cf + dh$

$$= \begin{pmatrix} \cdot & + & \cdot & \cdot \\ \cdot & \cdot & \cdot & + \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \qquad = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ + & \cdot & \cdot & \cdot \\ \cdot & \cdot & + & \cdot \end{pmatrix} \qquad = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & + & \cdot & \cdot \\ \cdot & \cdot & \cdot & + \end{pmatrix}$$

We will create 7 matrices in such a way that the above 4 matrices can be generated by addition and subtraction operations over these 7 matrices. Furthermore, the 7 matrices themselves can be produced by 7 multiplications and some additions and subtractions.

Reachability Queries

$$P_1 = A_1 \cdot B_1 = a \cdot (f - h) = af - ah \qquad P_2 = A_2 \cdot B_2 = (a + b) \cdot h = ah + bh$$

$$= \begin{pmatrix} \cdot & + & \cdot & - \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \qquad = \begin{pmatrix} \cdot & \cdot & \cdot & + \\ \cdot & \cdot & \cdot & + \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$s = af + bh$$

$$= \begin{pmatrix} \cdot & + & \cdot & \cdot \\ \cdot & \cdot & \cdot & + \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} = P_1 + P_2$$

$A_1 = a$, $B_1 = (f - h)$,
$A_2 = (a + b)$, $B_2 = h$,
$A_3 = (c + d)$, $B_3 = e$,
$A_4 = d$, $B_4 = (g - d)$,
$A_5 = (a + d)$, $B_5 = (e + h)$,
$A_6 = (b - d)$, $B_6 = (g + h)$,
$A_7 = (c - a)$ $B_7 = (e + f)$

Reachability Queries

$$P_1 = A_1 \cdot B_1 = a \cdot (f - h) = af - ah$$

$$= (a\ b\ c\ d) \begin{pmatrix} \cdot & + & \cdot & - \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix}$$

Reachability Queries

$$P_3 = A_3 \cdot B_3 = (c + d) \cdot e = ce + de \quad P_4 = A_4 \cdot B_4 = d \cdot (g - e) = dg - de$$

$$= \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ + & \cdot & \cdot & \cdot \\ + & \cdot & \cdot & \cdot \end{pmatrix} \qquad = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ - & \cdot & + & \cdot \end{pmatrix}$$

$$t = ce + dg$$

$$= \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ + & \cdot & \cdot & \cdot \\ \cdot & \cdot & + & \cdot \end{pmatrix} = P_3 + P_4$$

$A_1 = a,$ $B_1 = (f - h),$
$A_2 = (a + b),$ $B_2 = h,$
$A_3 = (c + d),$ $B_3 = e,$
$A_4 = d,$ $B_4 = (g - d),$
$A_5 = (a + d),$ $B_5 = (e + h),$
$A_6 = (b - d),$ $B_6 = (g + h),$
$A_7 = (c - a)$ $B_7 = (e + f)$

Reachability Queries

$P_5 = A_5 \cdot B_5 = (a + d) \cdot (e + h)$
$= ae + ah + de + dh$

$$= \begin{pmatrix} + & \cdot & \cdot & + \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ + & \cdot & \cdot & + \end{pmatrix}$$

$r = ae + bg$

$$= \begin{pmatrix} + & \cdot & \cdot & \cdot \\ \cdot & \cdot & + & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$= P_5 + P_4 - P_2 + P_6$

$P_6 = A_6 \cdot B_6 = (b - d) \cdot (g + h)$
$= bg + bh - dg - dh$

$$= \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & + & + \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & - & - \end{pmatrix}$$

$A_1 = a,$	$B_1 = (f - h),$
$A_2 = (a + b),$	$B_2 = h,$
$A_3 = (c + d),$	$B_3 = e,$
$A_4 = d,$	$B_4 = (g - d),$
$A_5 = (a + d),$	$B_5 = (e + h),$
$A_6 = (b - d),$	$B_6 = (g + h),$
$A_7 = (c - a)$	$B_7 = (e + f)$

Reachability Queries

$$P_5 + P_4 = \begin{pmatrix} + & \cdot & \cdot & + \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & + & + \end{pmatrix}$$

$$P_5 + P_4 - P_2 = \begin{pmatrix} + & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & - \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & + & + \end{pmatrix}$$

$$P_5 + P_4 - P_2 + P_6 = \begin{pmatrix} + & \cdot & \cdot & \cdot \\ \cdot & \cdot & + & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$P_7 = A_7 \cdot B_7 = (a - c) \cdot (e + f)$
$= ae + af - ce - cf$

$$= \begin{pmatrix} + & + & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ - & - & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

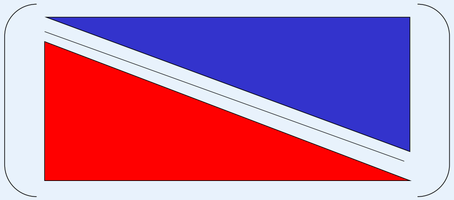
$A_1 = a,$ $B_1 = (f - h),$
$A_2 = (a + b),$ $B_2 = h,$
$A_3 = (c + d),$ $B_3 = e,$
$A_4 = d,$ $B_4 = (g - d),$
$A_5 = (a + d),$ $B_5 = (e + h),$
$A_6 = (b - d),$ $B_6 = (g + h),$
$A_7 = (c - a)$ $B_7 = (e + f)$

$u = cf + dh$

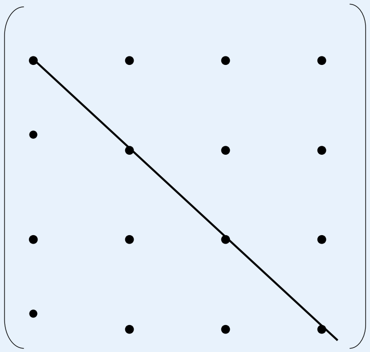
$$= \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & + & \cdot & \cdot \\ \cdot & \cdot & \cdot & + \end{pmatrix} = P_5 + P_1 - P_3 - P_7$$

Warren's Algorithm

Warren's algorithm is a quite simple way to generate a boolean matrix to represent the transitive closure of a graph *G*. Assume that *G* is represented by a boolean matrix *M* in which $M(i, j) = 1$ if edge (i, j) is in *G*, and $M(i, j) = 0$ if (i, j) is not in *G*. Then, the matrix *M'* for the transitive closure of *G* can be computed from *M*, in which $M'(i, j) = 1$ if there exits a path from *i* to *j* in *G*, and $M'(i, j) = 0$ if there is no path from *i* to *j* in *G*. Warren's algorithm is given below:



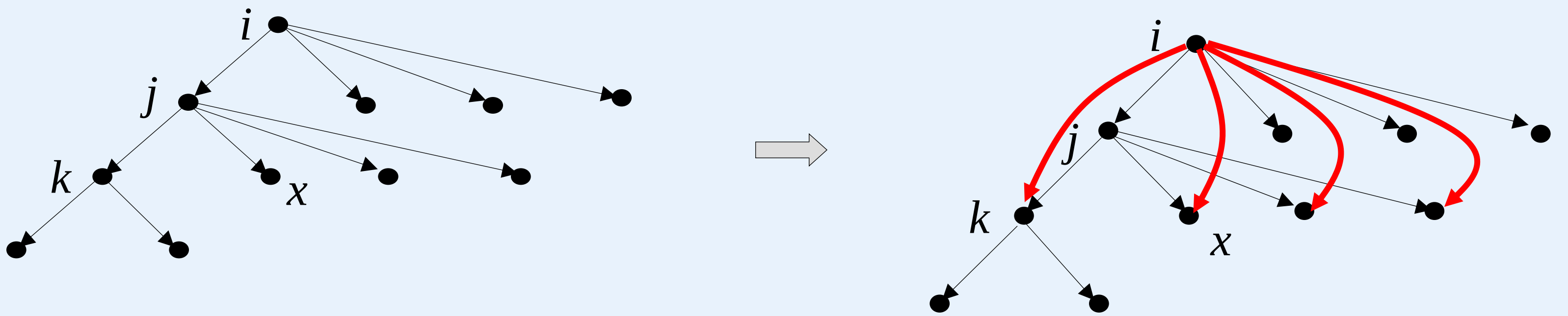
Algorithm *Warren*
for $i = 2$ to n **do**
    for $j = 1$ to $i - 1$ **do**
    {**if** $M(i, j) = 1$ **then** set $M(i, *) = M(i, *) \vee M(j, *)$;}
for $i = 1$ to $n - 1$ **do**
    for $j = i + 1$ to n **do**
    {**if** $M(i, j) = 1$ **then** set $M(i, *) = M(i, *) \vee M(j, *)$;}



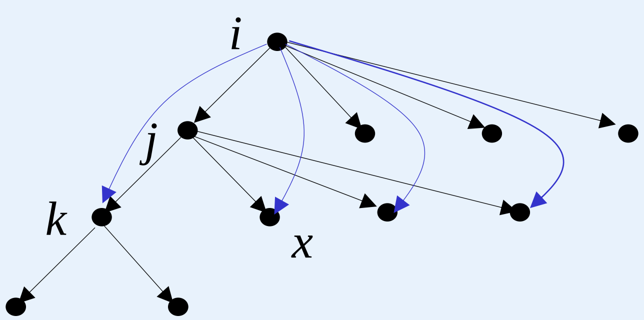
In the algorithm, $M(i, *)$ denotes row *i* of *M*.
The theoretic time complexity of Warren's algorithm is $O(n^3)$.

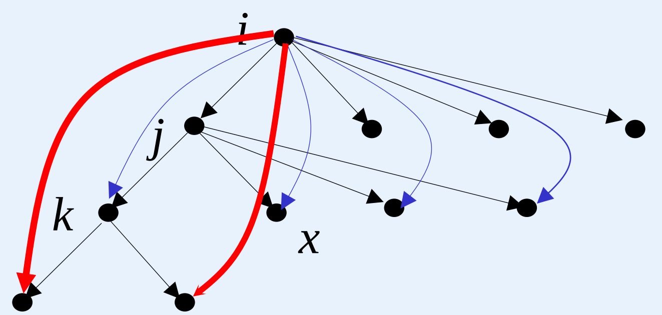
Reachability Queries

if $M(i, j) = 1$ **then** set $M(i, *) = M(i, *) \vee M(j, *)$



if $M(i, k) = 1$ **then** set $M(i, *) = M(i, *) \vee M(k, *)$





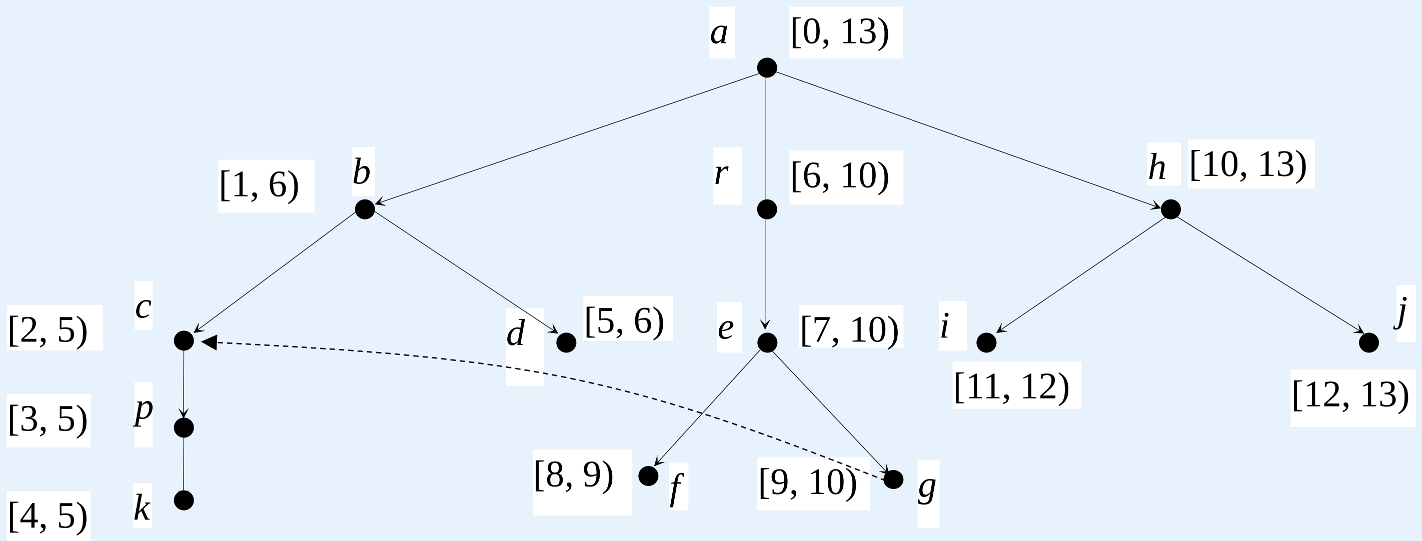
S. Warshall, “A Theorem on Boolean Matrices,” *JACM*, 9. 1(Jan. 1962), 11 - 12.
H.S. Warren, “A Modification of Warshall’s Algorithm for the Transitive Closure of Binary Relations,” *Commun. ACM* 18, 4 (April 1975), 218 - 220.

First kind of tree encoding

- **Definition**
  - We can assign each node v in a tree T an interval $[\alpha_v, \beta_v)$, where α_v is v's preorder number (denoted *pre*(v)) and β_v - 1 is equal to the largest preorder number among all the nodes in $T[v]$ (subtree rooted at v).
  - So another node u labeled $[\alpha_u, \beta_u)$ is a descendant of v (with respect to T) iff $\alpha_u \in [\alpha_v, \beta_v)$.
  - If $\alpha_u \in [\alpha_v, \beta_v)$, we say, $[\alpha_u, \beta_u)$ is subsumed by $[\alpha_v, \beta_v)$. This method is called the *tree labeling*.

H. Wang, H. He, J. Yang, P.S. Yu, and J. X. Yu, Dual Labeling: Answering Graph Reachability Queries in Constant time, in Proc. of Int. Conf. on Data Engineering, Atlanta, USA, April -8 2006.

Example:



For a directed graph, the intervals cannot be used to check reachability. The containment is just a sufficient condition, not a necessary condition.

Reachability checking based on tree encoding

Directed acyclic graphs (DAGs)

- Find a spanning tree T of $G(V, E)$, and assign each node v an interval.
- Examine all the nodes in G in a reverse topological order and do the following:

  For every edge (v, u), add all the intervals associated with the node u to the intervals associated with the node v.

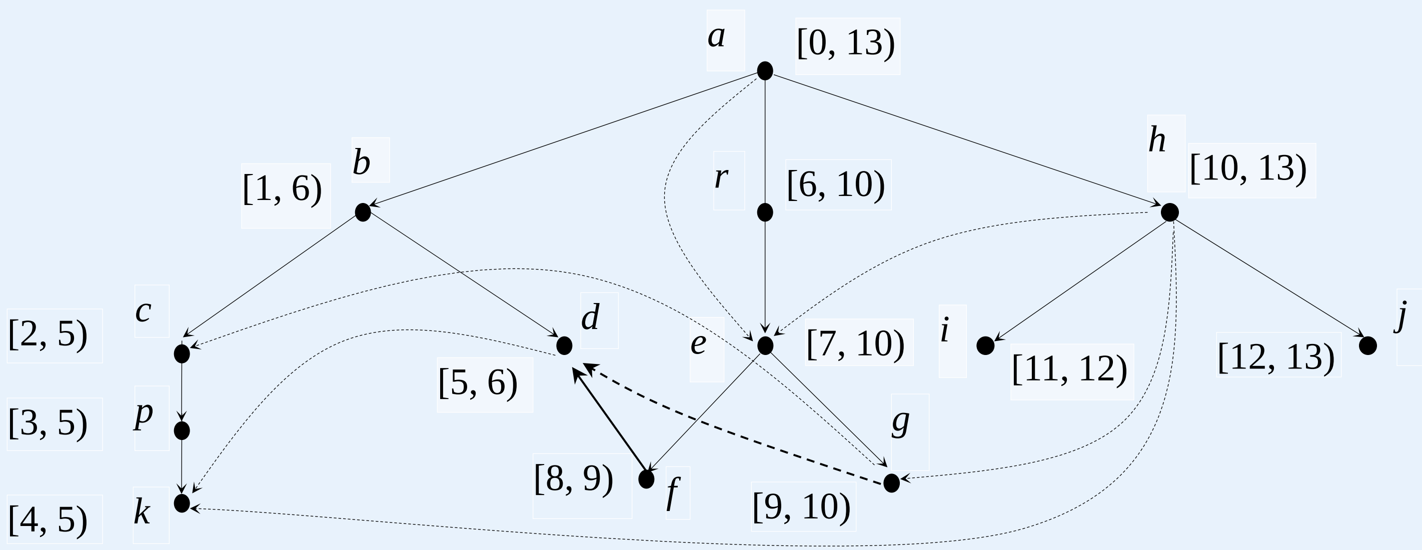
Topological order of a directed acyclic graph:

Linear ordering of the vertices of G such that if $(u, v) \in E$, then u appears somewhere before v.

Topological order of a directed acyclic graph:

Linear ordering of the vertices of G such that if $(u, v) \in E$, then u appears somewhere before v.

Example:



Topological order: *a*, *b*, *r*, *h*, *e*, *f*, *g*, *d*, *c*, *p*, *k*, *i*, *j*

When we navigate along a topological order, for every edge (v, u), add all the intervals associated with the node u to the intervals associated with the node v.

1. When adding an interval [i, j) to the interval sequence associated with a node, if an interval [i', j') is subsumed by [i, j), it will be discarded from the sequence. In other words: if $i' \in [i, j)$, then discard [i', j').
2. On the other hand, if an interval [i', j') is equal to [i, j) or subsumes [i, j). [i, j) will not be added to the sequence. Otherwise, [i, j) will be inserted.

Reverse topological order:

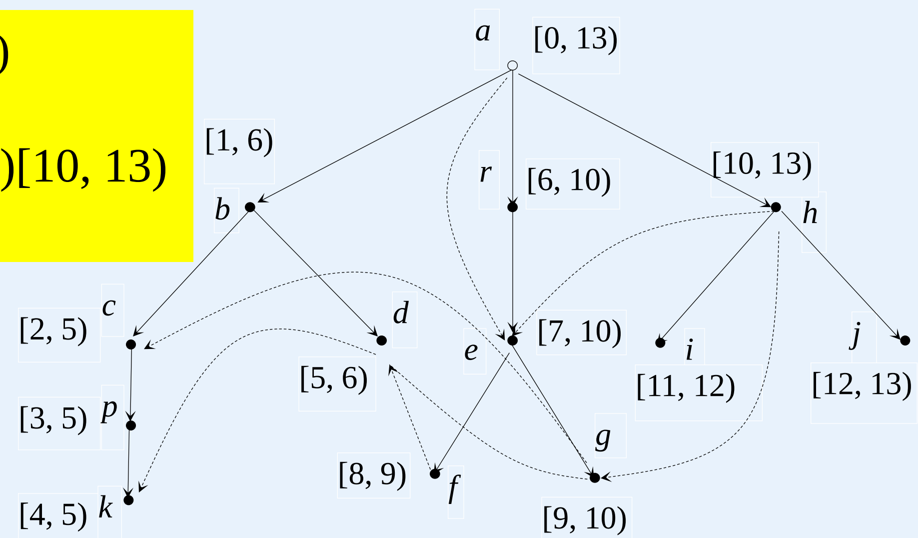
A sequence of the nodes of *G* such that for any edge (*u*, *v*) *v* appears before *u* in the sequence.

k, p, c, d, f, g, i, j, e, r, b, h, a

Reverse topological order

$L(k) = [4, 5)$
$L(p) = [3, 5)$
$L(c) = [2, 5)$
$L(d) = [4, 5)[5, 6)$
$L(f) = [4, 5)[5, 6)[8, 9)$
$L(g) = [2, 5)[5, 6)[9, 10)$
$L(i) = [11, 12)$
$L(j) = [12, 13)$
$L(e) = [2, 5)[5, 6)[7, 10)$

$L(r) = [2, 5)[5, 6)[6, 10)$
$L(b) = [1, 6)$
$L(h) = [2, 5)[5, 6)[7, 10)[10, 13)$
$L(a) = [10, 13)$



Generation of interval sequences

- Create interval sequences for all the nodes along the reverse topological order
- First of all, we notice that each leaf node is exactly associated with one interval, which is trivially sorted according to the first element in each interval.
- Let v_1, ..., v_l be the child nodes of v, associated with the interval sequences L_1, ..., L_l, respectively.
- Assume that the intervals in each L_i are sorted. We will merge all L_i's into the interval sequence L associated with v as follows.

  - Let $[a_1, b_1)$ (from L) and $[a_2, b_2)$ (from L_i) be the interval encountered. We will perform the following checkings:

- Let $[a_1, b_1)$ (from L) and $[a_2, b_2)$ (from L_i) be the interval encountered. We will perform the following checkings:

$L = \ldots [a_1, b_1) [a_1', b_1') \ldots$
$L_i = \ldots [a_2, b_2) [a_2', b_2') \ldots$

- **If** $a_2 >= a_1$ **then**
  {**if** $a_2 \in [a_1, b_1)$ **then** go to the interval next to $[a_2, b_2)$ and compare it with $[a_1, b_1)$ in a next step
  else go to the interval next to $[a_1, b_1)$ and compare it with $[a_2, b_2)$ in a next step.}
- **If** $a_1 > a_2$ **then**
  {**if** $a_1 \in [a_2, b_2)$ **then** remove $[a_1, b_1)$ from L and compare the interval next to $[a_1, b_1)$ with $[a_2, b_2)$ in a next step.
  else insert $[a_2, b_2)$ into L before $[a_1, b_1)$.}

Generation of the interval sequence for node *e*:

Initially, $L(e)$= [7, 10).

p, *q* are pointer variables used to scan $L(e)$ and $L(f)$, respectively.

First, merge $L(e)$ = [7, 10) with $L(f)$ = [4, 5)[5, 6)[8, 9).

p

$L(e)$ = [7, 10)

$L(f)$ = [4, 5)[5, 6)[8, 9)

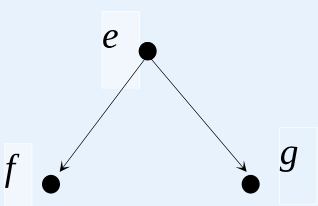
q

p

[4, 5)[7, 10)

[4, 5)[5, 6)[8, 9)

q



p

[4, 5)[5, 6)[7, 10)

[4, 5)[5, 6)[8, 9)

q

p

[4, 5)[5, 6)[7, 10)

[4, 5)[5, 6)[8, 9)

q

Secondly, merge $L'(e) = [4, 5)[5, 6)[7, 10)$ with $L(g) = [2, 5)[5, 6)[9, 10)$.

$$\begin{array}{c} p \\ \downarrow \\ L'(e) = [4, 5)[5, 6)[7, 10) \\ L(g) = [2, 5)[5, 6)[9, 10) \\ \uparrow \\ q \end{array} \Rightarrow \begin{array}{c} p \\ \downarrow \\ L'(e) = [5, 6)[7, 10) \\ L(g) = [2, 5)[5, 6)[9, 10). \\ \uparrow \\ q \end{array} \Rightarrow$$

$$\begin{array}{c} p \\ \downarrow \\ L'(e) = [2, 5)[5, 6)[7, 10) \\ L(g) = [2, 5)[5, 6)[9, 10) \\ \uparrow \\ q \end{array} \Rightarrow \begin{array}{c} p \\ \downarrow \\ L'(e) = [4, 5)[5, 6)[7, 10) \\ L(g) = [2, 5)[5, 6)[9, 10) \\ \uparrow \\ q \end{array}$$

Secondly, merge $L'(e)$ = [4, 5)[5, 6)[7, 10) with $L(g)$ = [2, 5)[5, 6)[9, 10).

$$\begin{array}{ll} & \qquad\qquad\qquad\qquad\quad p \\ & \qquad\qquad\qquad\qquad\quad \downarrow \\ L'(e) = & [2, 5)[5, 6)[7, 10) \\ L(g) = & [2, 5)[5, 6)[9, 10) \\ & \qquad\qquad\qquad\qquad\quad \uparrow \\ & \qquad\qquad\qquad\qquad\quad q \end{array} \qquad\qquad \begin{array}{ll} & \qquad\qquad\qquad\qquad\quad p \\ & \qquad\qquad\qquad\qquad\quad \downarrow \\ L'(e) = & [2, 5)[5, 6)[7, 10) \\ L(g) = & [2, 5)[5, 6)[9, 10) \\ & \qquad\qquad\qquad\qquad\qquad\qquad\quad \uparrow \\ & \qquad\qquad\qquad\qquad\qquad\qquad\quad q \end{array}$$

Obviously, $|L| \leq b$ (the number of the leaf nodes in the spanning tree T) and the intervals in L are sorted. The time spent on this process is $O(d_v b)$, where d_v represents the outdegree of v. So the whole cost is bounded by

$$O(\sum_v d_v b\ ) = O(be).$$

Here, e is the *number* of edges in the graph. We have

$$O(\sum_v d_v) = e.$$

The size of the data structure is bounded by $O(bn)$.

Reachability checking for DAGs

- Let u and v be two nodes of G.
- u is a descendant of v iff there exists an interval $[\alpha, \beta)$ in $L(v)$ such that $\alpha_u \in [\alpha, \beta)$.

Example:

$[\alpha_k, \beta_k) = [4, 5)$
$L(r) = [2, 5)[5, 6)[6, 10)$

⇨

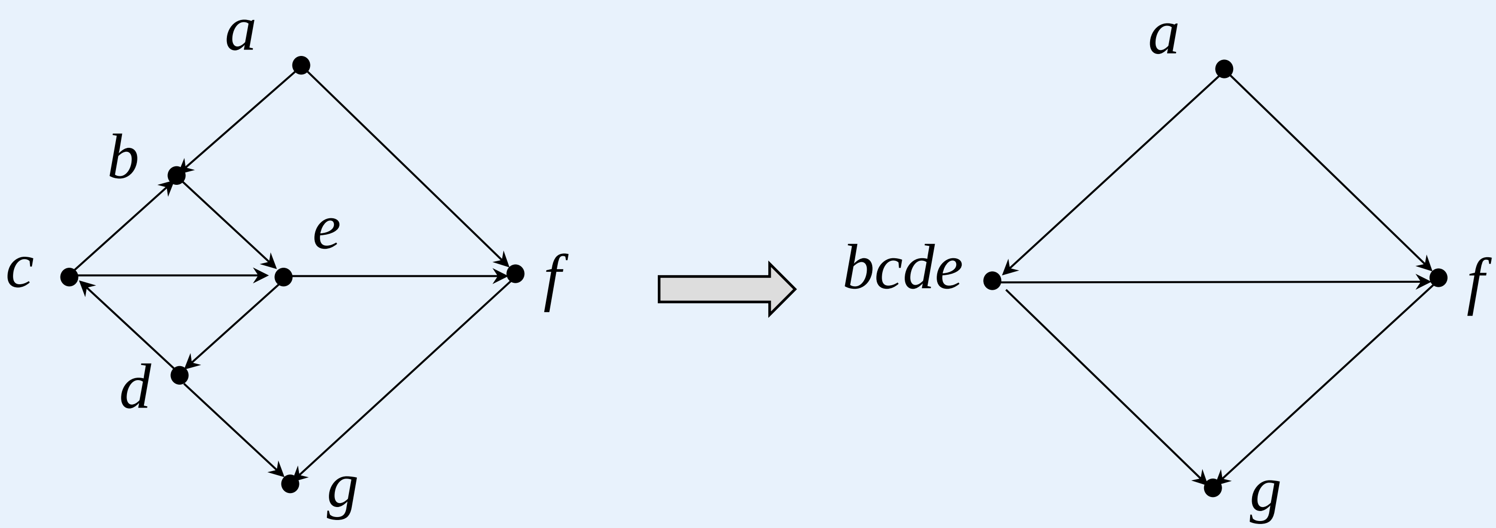
Node k is a descendant of node r.

Reachability checking for cyclic graphs

- Using the Tarjan's algorithm to recognize all the *strongly connected components* (*SCCs*). In each SCC, any two nodes are reachable from each other.
- Collapse each SCC to a single node. In this way, any cyclic graph *G* is transformed to a DAG *G'*.
- Let *u* and *v* be to two nodes in *G*. Check their reachability according to two cases:
  • *u* and *v* are in the same SCC.
  • *u* and *v* are in two different SCC.

R. Tarjan: Depth-first Search and Linear Graph Algorithms, SIAM J. Compt. Vol. 1. No. 2. June 1972, pp. 146-140.

Reachability Queries



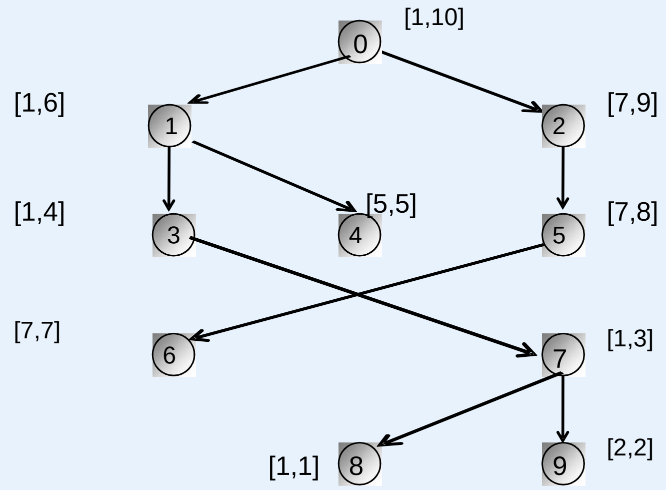
Second kind of tree encoding: Using tree encoding as a filter

- Each node v in a tree T is labeled with a with a range: $I_v = [r_x, r_v]$, where r_v is the postorder number of v (the postorder numbers are assumed to begin at 1) and r_x is the lowest postorder number of any node x in the subtree $T[v]$ rooted at v (also, including v).
- This approach guarantees that the containment between intervals is equivalent to the reachability relationship between the nodes, since the postorder traversal enters a node before all of its descendants have been visited. In other words,

$$u \rightsquigarrow v \Leftrightarrow I_v \subseteq I_u.$$

H. Yildirim, V. Chaoji, and M. J. Zaki, "GRAIL: Scalable reachabil1473 ity index for llarge graphs," in *Proc. VLDB Endowment*, vol. 3, no. 1, 1474 pp. 276–284, 2010.

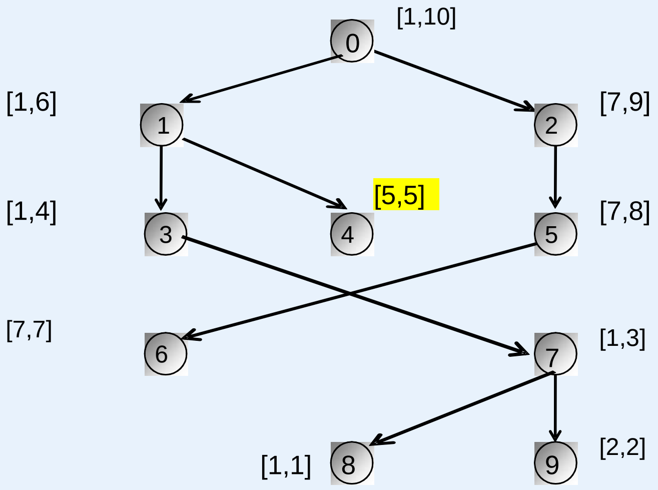
Example:

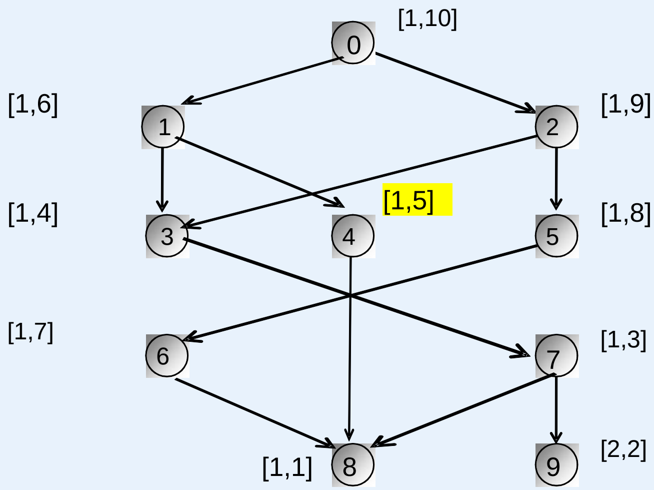


The above figure shows the interval labeling on a tree, assuming that the children are ordered from left to right. It is easy to see that reachability can be answered by interval containment. For example, $1 \rightsquigarrow 9$, since $I_9 = [2, 2] \subset [1, 6] = I_1$, but $2 \not\rightsquigarrow 7$, since $I_7 = [1, 3] \not\subset [7, 9] = I_2$.

Using tree encoding as a filter

To generalize the interval labeling to a DAG *G*, we have to ensure that a node is not visited more than once during a bottom-up search of *G*, and a node will keep the postorder number r_v of its first visit. Its r_x is now the lowest postorder number in the sub-graph rooted at *v*.

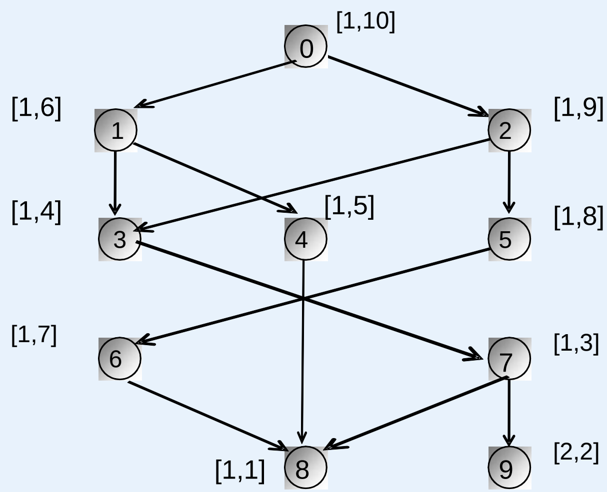




Reachability Queries

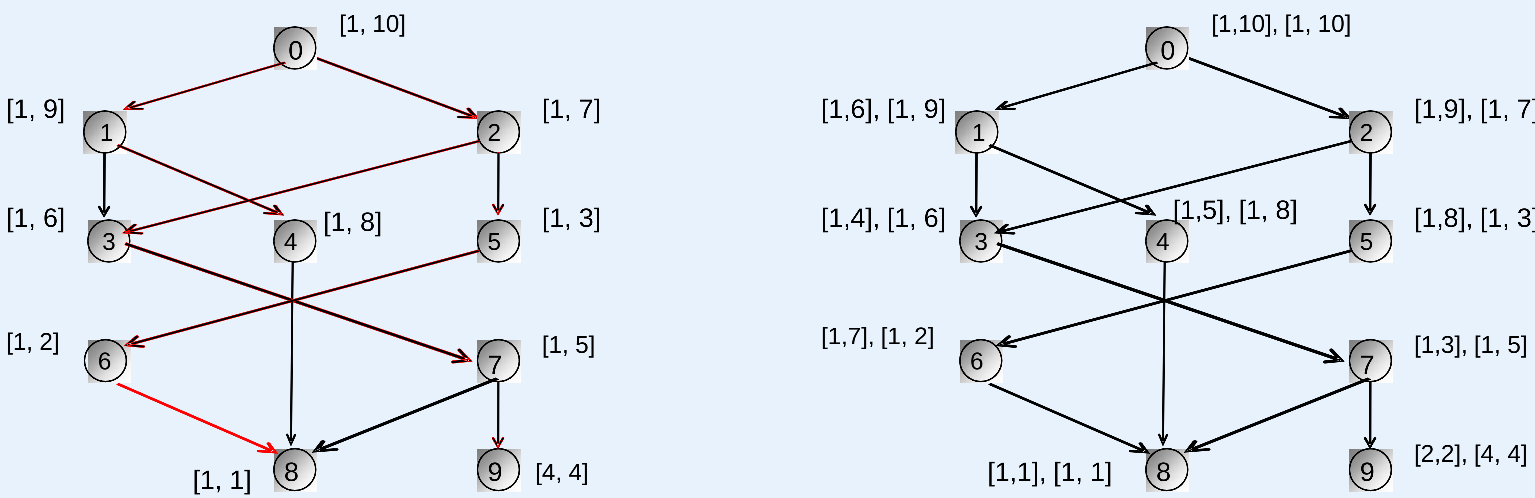
The above shows an interval labeling on a DAG, assuming a left to right ordering of the children. As one can see, interval containment of nodes in a DAG is not exactly equivalent to reachability.

For example, $5 \not\rightsquigarrow 4$, but $I_4 = [1, 5] \subseteq [1, 8] = I_5$. In other words, $I_v \subseteq I_u$ does not imply that $u \rightsquigarrow v$. On the other hand, one can show that $I_v \not\subset I_u \Rightarrow u \not\rightsquigarrow v$. (So the containment is a necessary condition, not a sufficient condition.)



Reachability Queries

- Instead of using a single interval, one can employs multiple intervals that are obtained via random graph traversals.
- We use the symbol *d* to denote the number of intervals to keep per node, which also corresponds to the number of graph traversals used to obtain the label.
- The following figure shows a DAG labeling using 2 intervals (the first interval assumes a left-to-right ordering of the children, whereas the second interval assumes a right-to-left ordering).



Index construction

An interval I_u^i is denoted as

$$I_u^i = [I_u^i[1], I_u^i[2]\ ] = [r_x, r_u]$$

Algorithm 1: Randomized Intervals

RandomizedLabeling(G, d):

foreach $i \leftarrow 1$ to d **do** (*d – number of intervals for each node*)

  $r \leftarrow 1$ // global variable: postorder number of node

  $Roots \leftarrow \{n : n \in roots(G)\}$

  foreach $x \in$ *Roots in random order* **do**

    Call **RandomizedVisit**(x, i, G)

RandomizedVisit(x, i, G) :

if *x visited before* **then return**

foreach $y \in Children(x)$ *in random order* **do**

  Call **RandomizedVisit**(y, i, G)

$r_c^* \leftarrow min\{I_c^i[1] : c \in Children(x)\}$

$I_x^i \leftarrow [min(r, r_c^*\ ), r]$ (*Compute $[I_x^i[1], I_x^i[2]\ ]$.*)

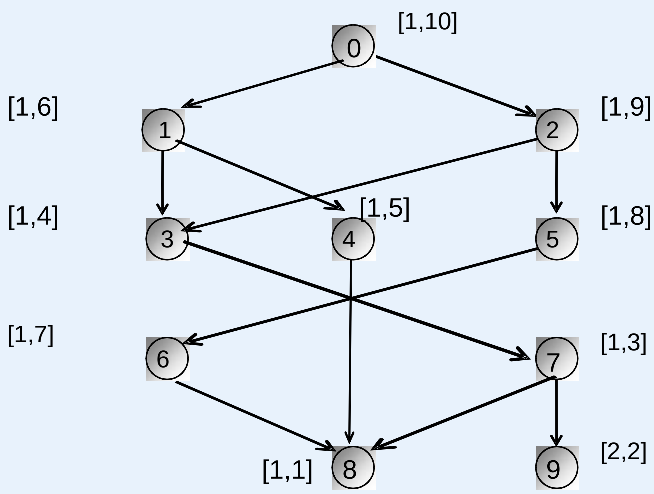
$r \leftarrow r + 1$

Reachability queries

- Assume that each node is associated with an single interval.
- To answer reachability queries between two nodes, u and v, we will first check whether $I_v \not\subseteq I_u$. If so, we can immediately conclude that $u \not\rightsquigarrow v$.
- On the other hand, if $I_v \subseteq I_u$, nothing can be concluded immediately since we know that the index can have false positives, i.e., exceptions. In this case, a DFS (depth-first search) is conducted, with recursive containment check based pruning, to answer queries. In the worst case, it needs O(n) time.
- Another way is to check the exception lists associated with the nodes:

$$E_x = \{y : (x, y) \text{ is an exception, i.e., } I_y \subseteq I_x \text{ and } x \not\rightsquigarrow y\}.$$

Reachability Queries



Exception lists:

$E_2 = \{1, 4\}$
$E_4 = \{3, 7, 9\}$
$E_5 = \{1, 3, 4, 7, 9\}$
$E_6 = \{1, 3, 4, 7, 9\}$

DFS with prunning

Algorithm 2: Reachability Testing (*for the case of only one interval*)

Reachable(u, v, G):

if $I_v \not\subset I_u$ **then**
  return *False* (* $u \not\rightsquigarrow v$ *)
else if use exception lists **then**
  if $v \in E_u$ **then return** *False* (* $u \not\rightsquigarrow v$ *)
  else return *True* (* $u \rightsquigarrow v$ *)
else (*No exception list. DFS with pruning using intervals.*)
  foreach $c \in$ *Children*(u) *such that* $I_v \subseteq I_c$ **do**
    if Reachable(c, v, G) **then**
      return *True* (* $u \rightsquigarrow v$ *)
  return *False* (* $u \not\rightsquigarrow v$ *)