Elementary Graph Algorithms

- Graph representation
- Graph traversal
  - Breadth-first search
  - Depth-first search
- Parenthesis theorem

Graphs

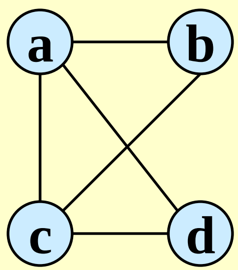
- *Graph G* = (V, E)
  - » V = set of vertices
  - » E = set of edges $\subseteq (V \times V)$
- Types of graphs
  - » Undirected: edge $(u, v) = (v, u)$; for all v, $(v, v) \notin E$ (No self loops.)
  - » Directed: (u, v) is edge from u to v, denoted as $u \rightarrow v$. Self loops are allowed.
  - » Weighted: each edge has an associated weight, given by a weight function $w : E \rightarrow \mathbf{R}$.
  - » Dense: $|E| \approx |V|^2$.
  - » Sparse: $|E| << |V|^2$.
- $|E| = O(|V|^2)$

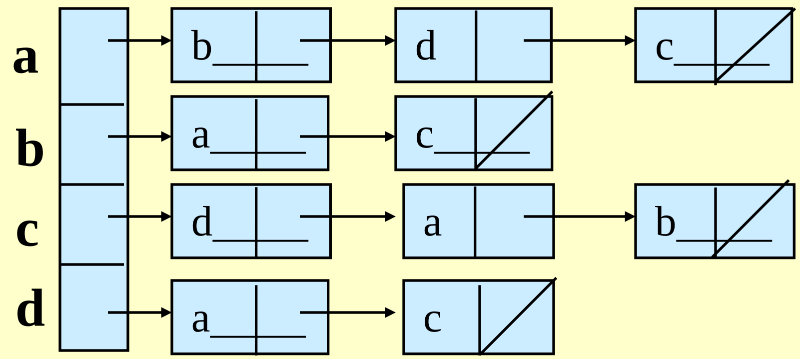
Graphs

- If $(u, v) \in E$, then vertex v is adjacent to vertex u.
- Adjacency relationship is:
  - » Symmetric if G is undirected.
  - » Not necessarily so if G is directed.
- If G is connected:
  - » There is a path between every pair of vertices.
  - » $|E| \geq |V| - 1$.
  - » Furthermore, if $|E| = |V| - 1$, then G is a tree.
- Other definitions in Appendix B (B.4 and B.5) as needed.

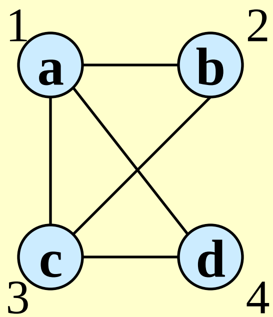
Representation of Graphs

- Two standard ways.
  - » Adjacency Lists.





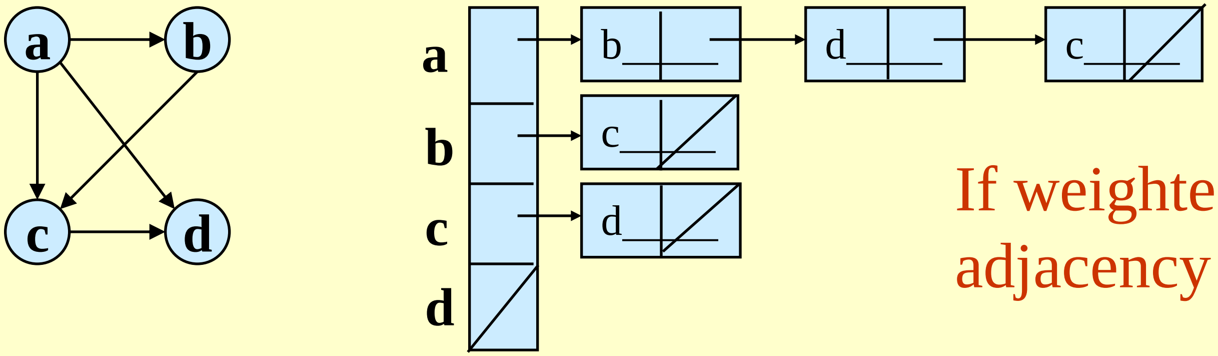
  - » Adjacency Matrix.



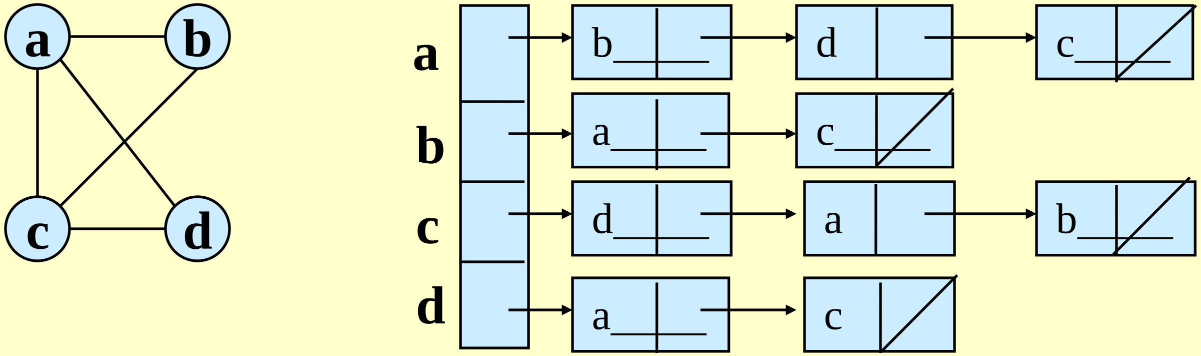
	1	2	3	4
1	0	1	1	1
2	1	0	1	0
3	1	1	0	1
4	1	0	1	0

Adjacency Lists

- Consists of an array *Adj* of |*V*| lists.
- One list per vertex.
- For $u \in V$, $Adj[u]$ consists of all vertices adjacent to u.



If weighted, store weights also in adjacency lists.



Storage Requirement

- For directed graphs:
  - Sum of lengths of all adj. lists is

$$\sum_{v \in V} \text{out-degree}(v) = |E|$$

No. of edges leaving v

  - Total storage: $\Theta(|V| + |E|)$
- For undirected graphs:
  - Sum of lengths of all adj. lists is

$$\sum_{v \in V} \text{degree}(v) = 2|E|$$

No. of edges incident on v. Edge (u,v) is incident on vertices u and v.

  - Total storage: $\Theta(|V| + |E|)$

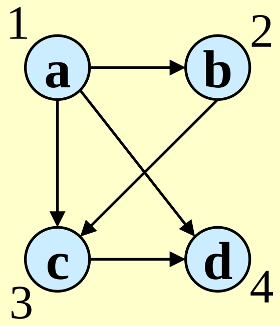
Pros and Cons: adj list

- Pros
  - » Space-efficient, when a graph is sparse.
  - » Can be modified to support many graph variants.
- Cons
  - » Determining if an edge $(u, v) \in G$ is not efficient.
    - Have to search in u's adjacency list. $\Theta(\text{degree}(u))$ time.
    - $\Theta(V)$ in the worst case.

Adjacency Matrix

- $|V| \times |V|$ matrix A.
- Number vertices from 1 to $|V|$ in some arbitrary manner.
- A is then given by:

$$A[i,j] = a_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$



	1	2	3	4
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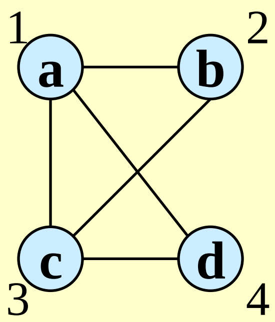
2	0	0	1	0
3	0	0	0	1
4	0	0	0	0



	1	2	3	4
1	0	1	1	1
2	1	0	1	0
3	1	1	0	1
4	1	0	1	0

$A = A^{\mathrm{T}}$ for undirected graphs.

Space and Time

- **Space:** $\Theta(V^2)$.
  - » Not memory efficient for large graphs.
- **Time:** to list all vertices adjacent to u: $\Theta(V)$.
- **Time:** to determine if $(u, v) \in E$: $\Theta(1)$.
- Can store weights instead of bits for weighted graph.

Graph-searching Algorithms

- Searching a graph:
  - » Systematically follow the edges of a graph to visit the vertices of the graph.
- Used to discover the structure of a graph.
- Standard graph-searching algorithms.
  - » Breadth-first Search (BFS).
  - » Depth-first Search (DFS).

Breadth-first Search

- **Input:** Graph $G = (V, E)$, either directed or undirected, and ***source vertex*** $s \in V$.
- **Output:**
  - » $d[v] =$ distance (smallest # of edges, or shortest path) from s to v, for all $v \in V$. $d[v] = \infty$ if v is not reachable from s.
  - » $\pi[v] = u$ such that (u, v) is last edge on shortest path $s \rightsquigarrow v$.
    - u is v's predecessor.
  - » Builds breadth-first tree with root s that contains all reachable vertices.

Definitions:

Path between vertices u and v: Sequence of vertices $(v_1, v_2, \ldots, v_k)$ such that $u=v_1$ and $v=v_k$, and $(v_i,v_{i+1}) \in E$, for all $1\le i \le k-1$.

Length of the path: Number of edges in the path.

Path is simple if no vertex is repeated.

Breadth-first Search

- Expands the frontier between discovered and undiscovered vertices uniformly across the breadth of the frontier.
  - » A vertex is "discovered" the first time it is encountered during the search.
  - » A vertex is "finished" if all vertices adjacent to it have been discovered.
- Colors the vertices to keep track of progress.
  - » White – Undiscovered.
  - » Gray – Discovered but not finished.
  - » Black – Finished.

BFS(G,s)

```
for each vertex u in V[G] – {s}
        do color[u] ← white
           d[u] ← ∝
           π[u] ← nil
color[s] ← gray
d[s] ← 0
π[s] ← nil
Q ← Φ
enqueue(Q,s)
while Q ≠ Φ
        do u ← dequeue(Q)
                for each v in Adj[u]
                        do if color[v] = white
                                then color[v] ← gray
                                     d[v] ← d[u] + 1
                                     π[v] ← u
                                     enqueue(Q,v)
                color[u] ← black
```

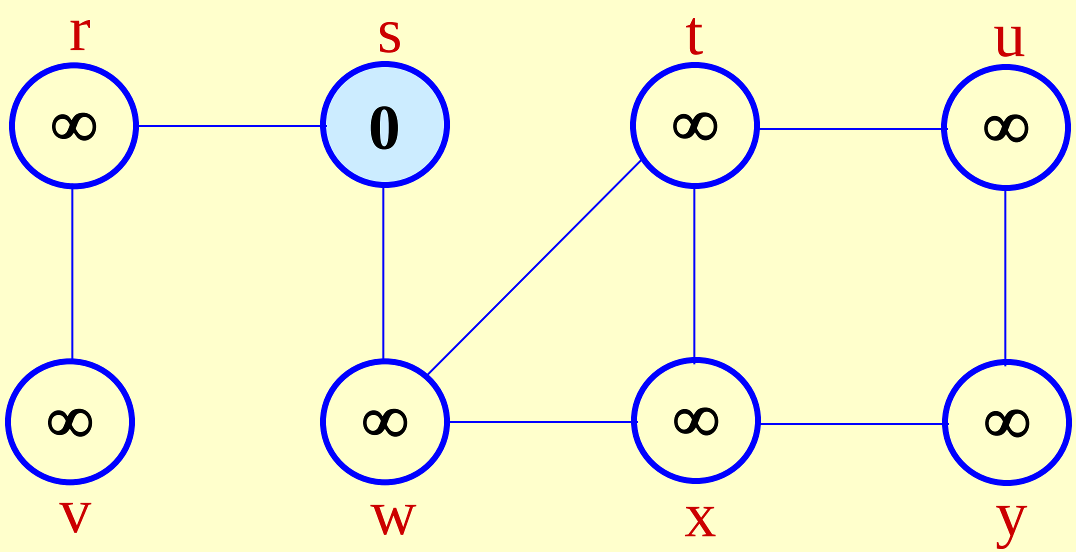
initialization

access source *s*

white: undiscovered
gray: discovered
black: finished

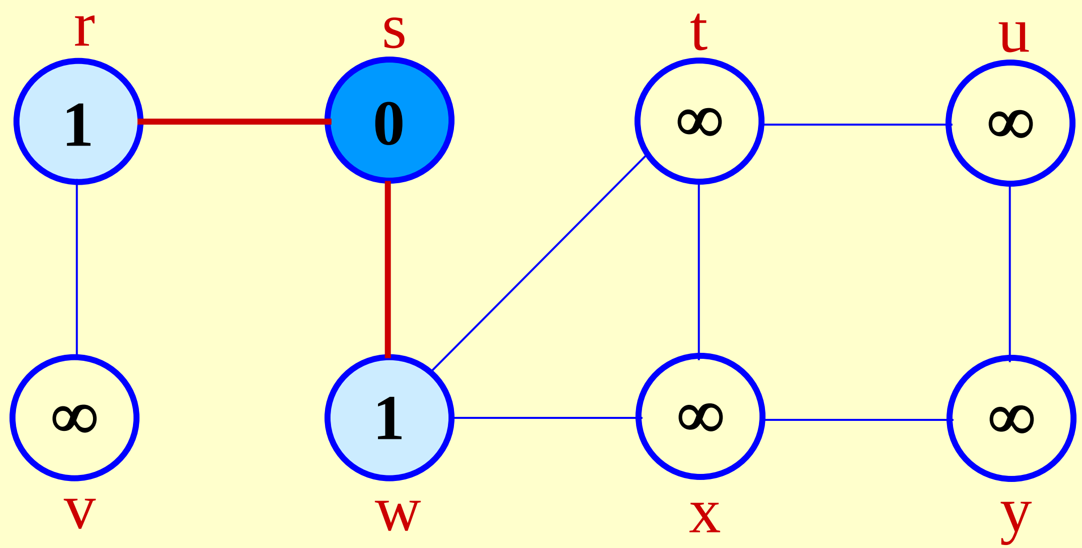
Q: a queue of discovered vertices
color[*v*]: color of v
d[*v*]: distance from s to v
π[*u*]: predecessor of v

Example (BFS)



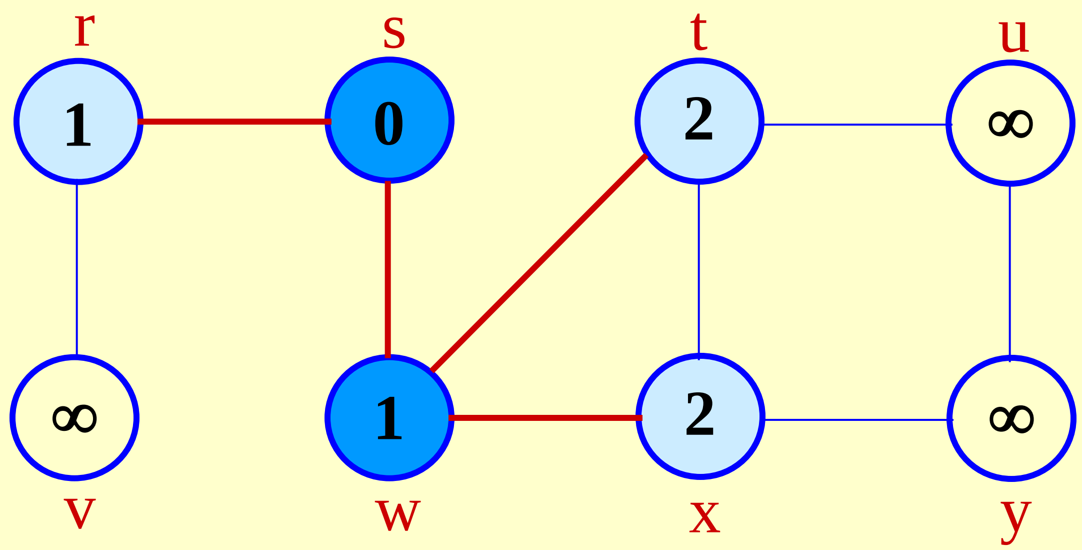
Q:	s
	0

Example (BFS)



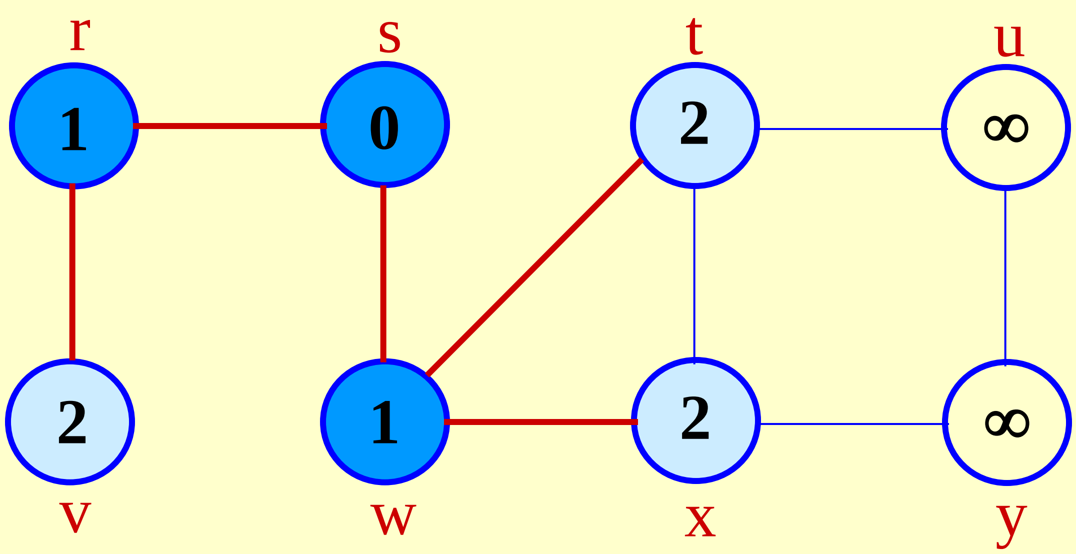
Q:	w	r
	1	1

Example (BFS)



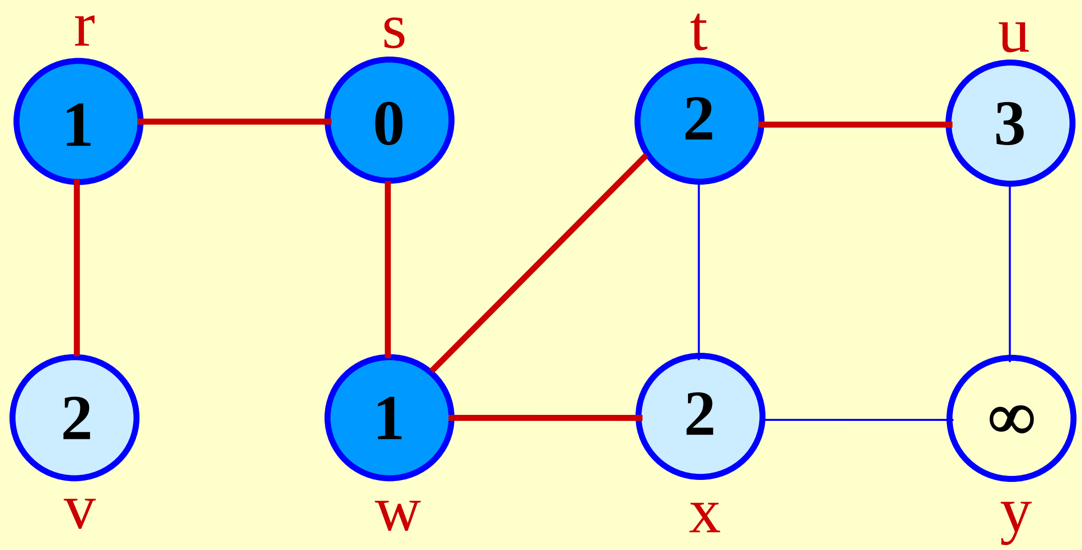
Q:	r	t	x
	1	2	2

Example (BFS)



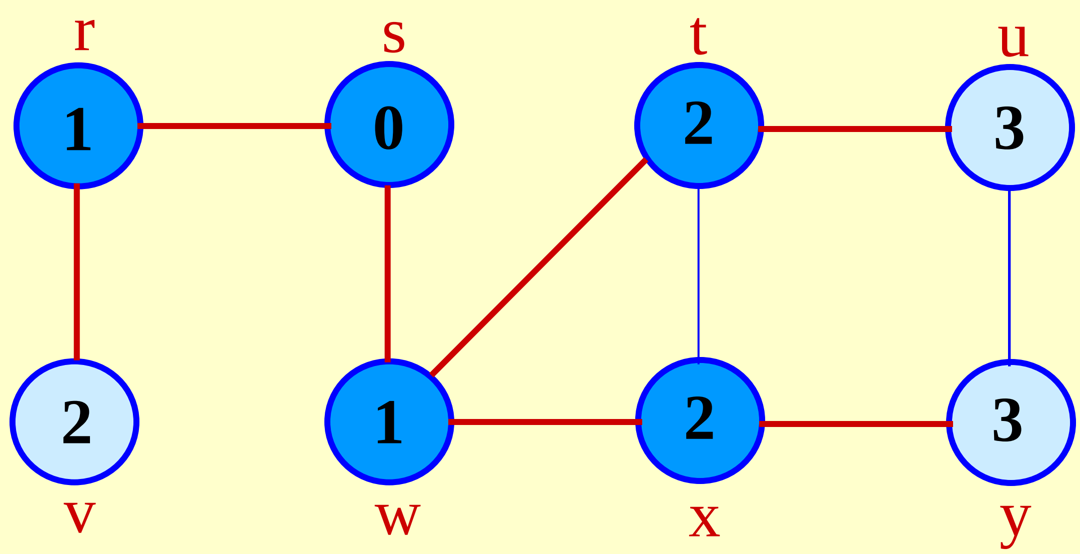
Q:	t	x	v
	2	2	2

Example (BFS)



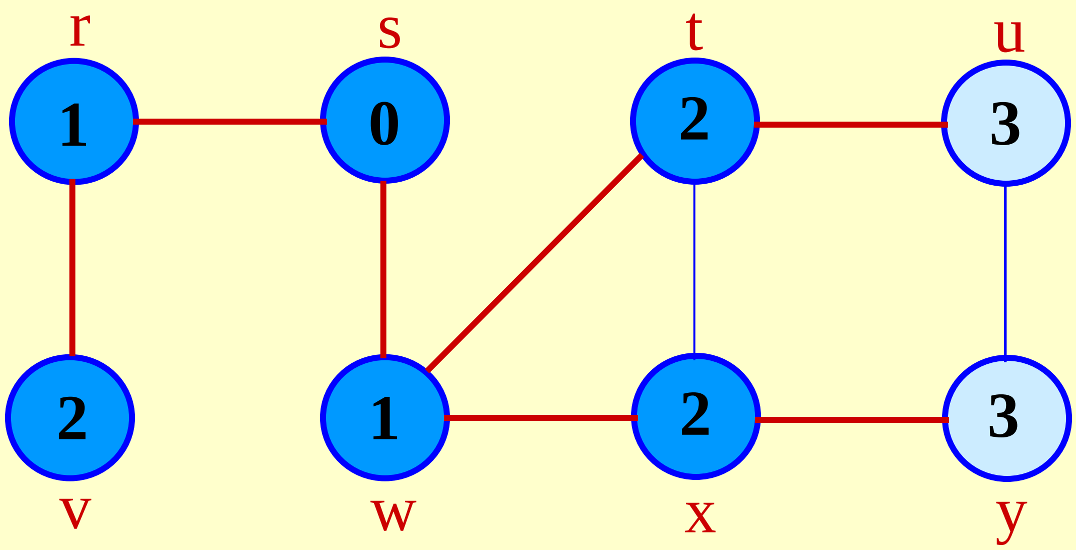
Q: x v u
2 2 3

Example (BFS)



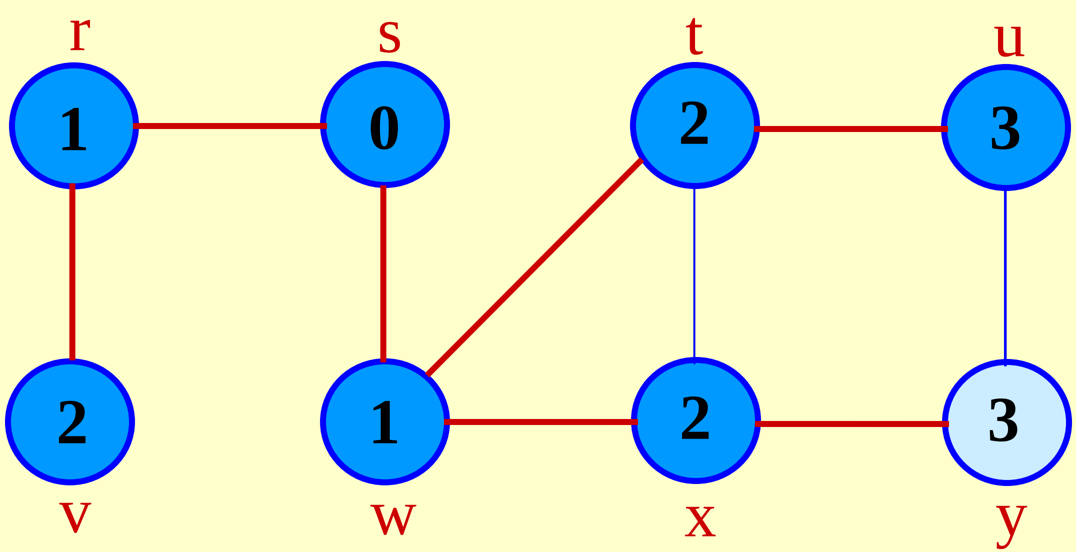
Q: v u y
2 3 3

Example (BFS)



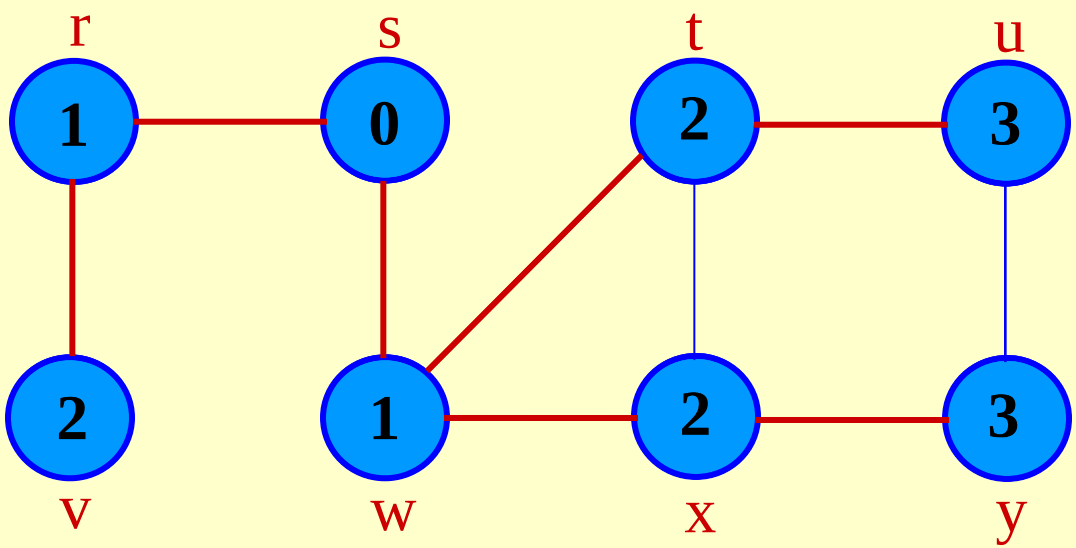
Q: u y
3 3

Example (BFS)



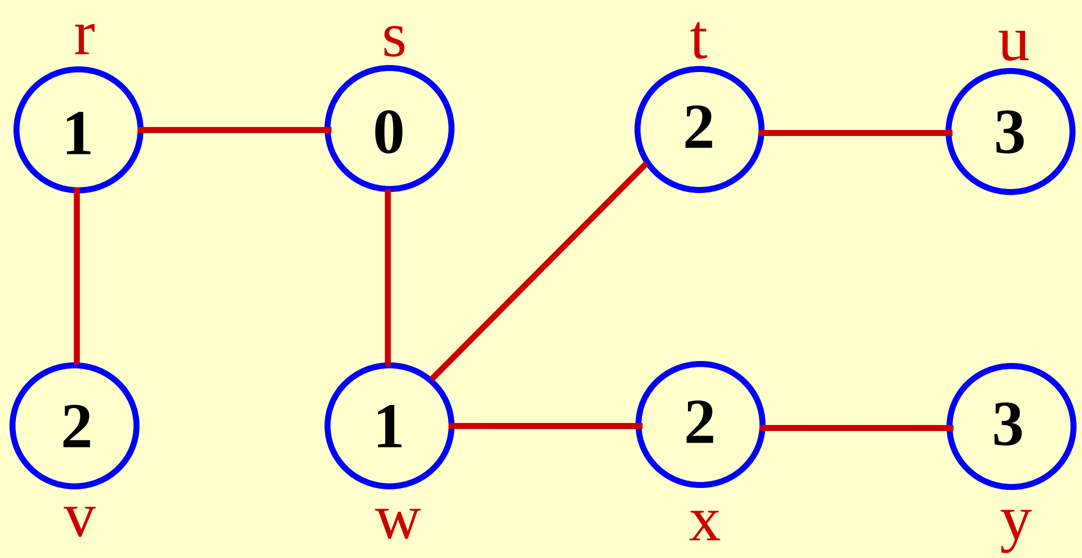
Q: y
3

Example (BFS)



Q: $\emptyset$

Example (BFS)



BF Tree

Analysis of BFS

- Initialization takes $O(|V|)$.
- Traversal Loop
  - » After initialization, each vertex is enqueued and dequeued at most once, and each operation takes $O(1)$. So, total time for queuing is $O(|V|)$.
  - » The adjacency list of each vertex is scanned at most once. The sum of lengths of all adjacency lists is $\Theta(|E|)$.
- Summing up over all vertices => total running time of BFS is $O(|V| + |E|)$, linear in the size of the adjacency list representation of graph.
- Correctness Proof
  - » We omit for BFS and DFS.
  - » Will do for later algorithms.

Breadth-first Tree

- For a graph $G = (V, E)$ with source s, the **predecessor subgraph** of G is $G_\pi = (V_\pi, E_\pi)$ where
  - » $V_\pi = \{v \in V : \pi[v] \neq nil\} \cup \{s\}$
  - » $E_\pi = \{(\pi[v], v) \in E : v \in V_\pi - \{s\}\}$
- The predecessor subgraph G_π is a **breadth-first tree** if:
  - » V_π consists of the vertices reachable from s and
  - » for all $v \in V_\pi$, there is a unique simple path from s to v in G_π that is also a shortest path from s to v in G.
- The edges in E_π are called **tree edges**.
  $|E_\pi| = |V_\pi| - 1$.

Depth-first Search (DFS)

- Explore edges out of the most recently discovered vertex v.
- When all edges of v have been explored, backtrack to explore other edges leaving the vertex from which v was discovered (its *predecessor*).
- "Search as deep as possible first."
- Continue until all vertices reachable from the original source are discovered.
- If any undiscovered vertices remain, then one of them is chosen as a new source and search is repeated from that source.

Depth-first Search

- **Input:** $G = (V, E)$, directed or undirected. No source vertex given!
- **Output:**
  - » 2 **timestamps** on each vertex. Integers between 1 and 2|V|.
    - $d[v]$ = ***discovery time*** (v turns from white to gray)
    - $f[v]$ = ***finishing time*** (v turns from gray to black)
  - » $\pi[v]$: predecessor of $v = u$, such that v was discovered during the scan of u's adjacency list.
- Coloring scheme for vertices as BFS. A vertex is
  - » "discovered" the first time it is encountered during the search.
  - » A vertex is "finished" if it is a leaf node or all vertices adjacent to it have been finished.

Pseudo-code

DFS(G)

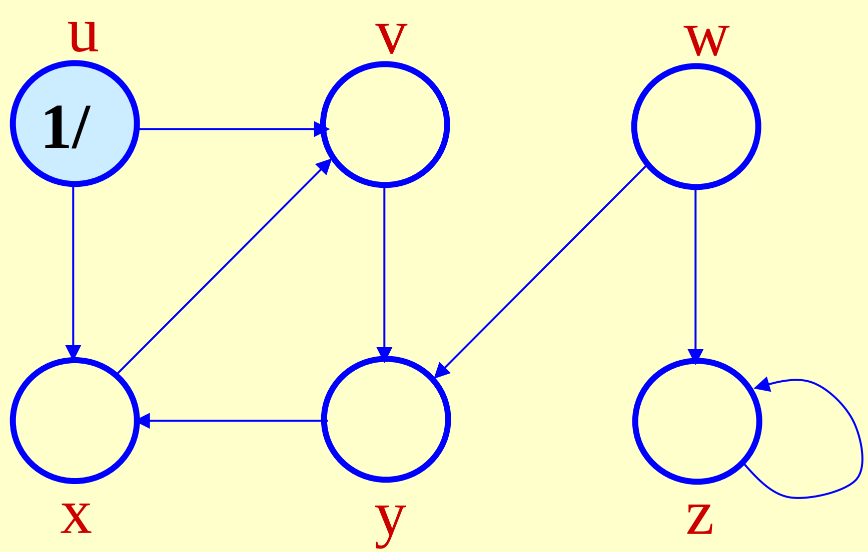
for each vertex $u \in V[G]$
  do *color*[u] $\leftarrow$ white
    $\pi[u] \leftarrow$ NIL
time $\leftarrow 0$
for each vertex $u \in V[G]$
  do if *color*[u] = white
    then DFS-Visit(u)

Uses a global timestamp ***time***.

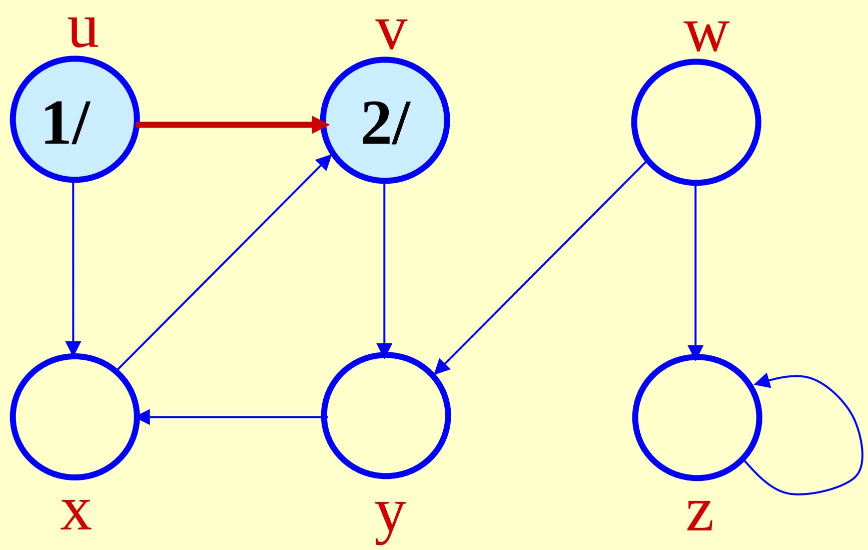
DFS-Visit(u)

color[u] $\leftarrow$ GRAY // White vertex u has been discovered
time $\leftarrow$ *time* + 1
$d[u] \leftarrow$ *time*
for each $v \in Adj[u]$
  do if *color*[v] = WHITE
    then $\pi[v] \leftarrow u$
      DFS-Visit(v)
color[u] $\leftarrow$ BLACK // Blacken u; it is finished.
$f[u] \leftarrow$ *time* $\leftarrow$ *time* + 1

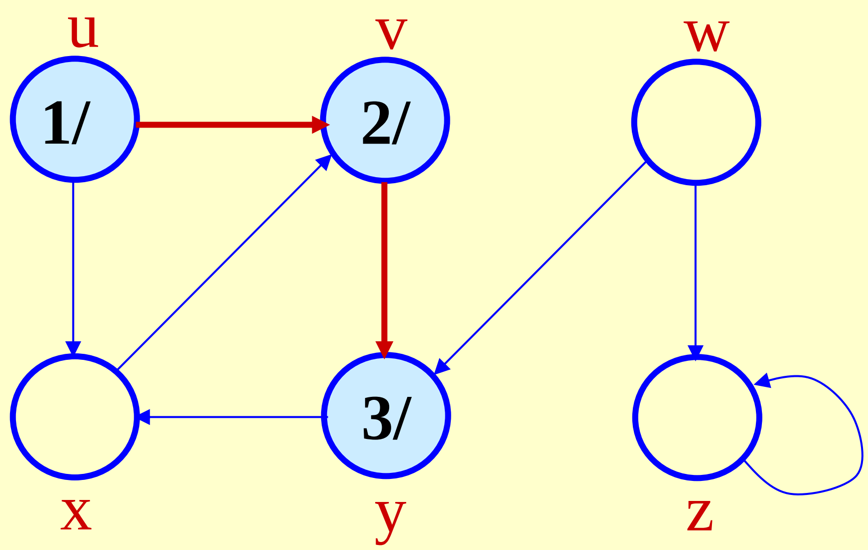
Example (DFS)



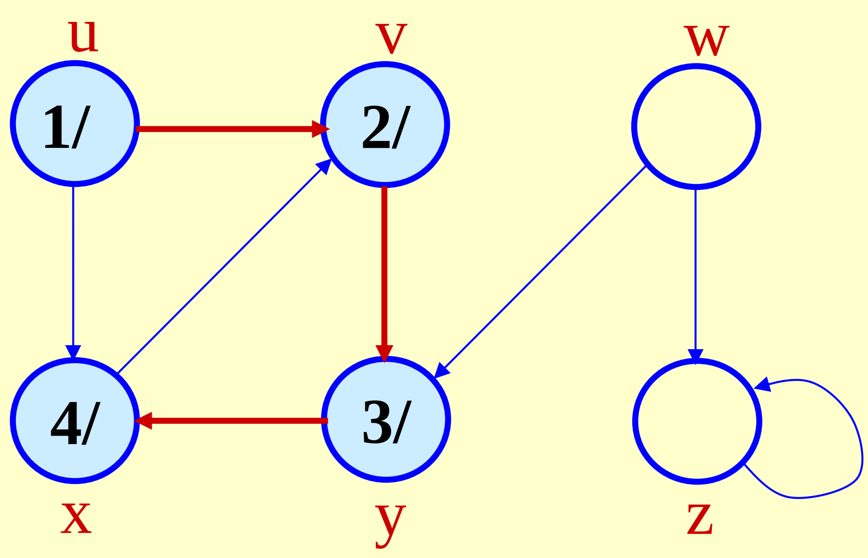
Example (DFS)



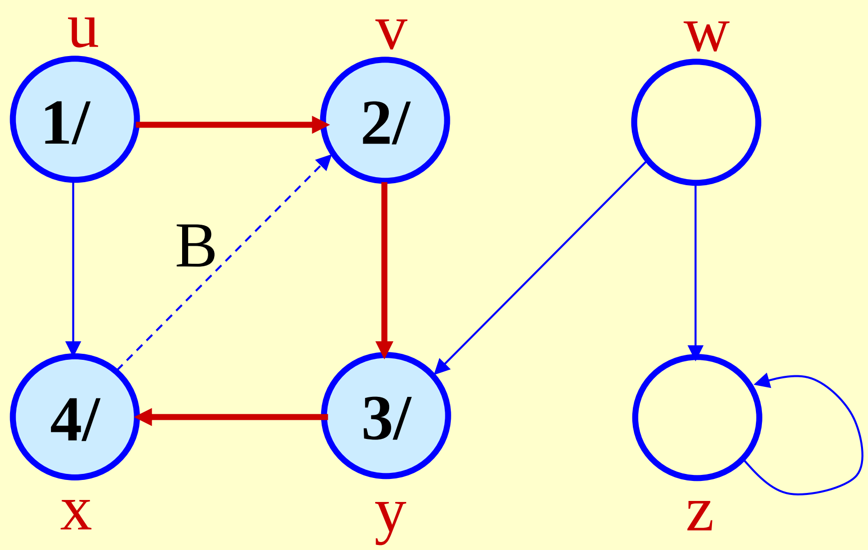
Example (DFS)



Example (DFS)



Example (DFS)



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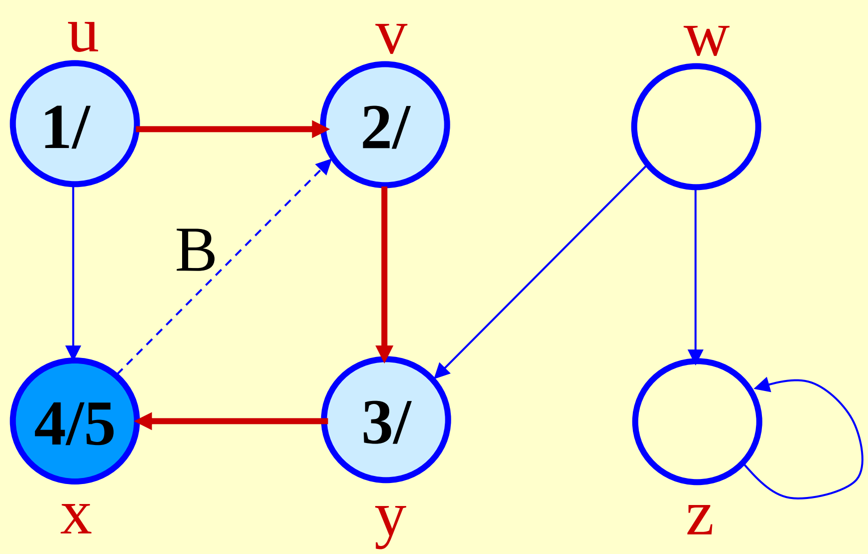
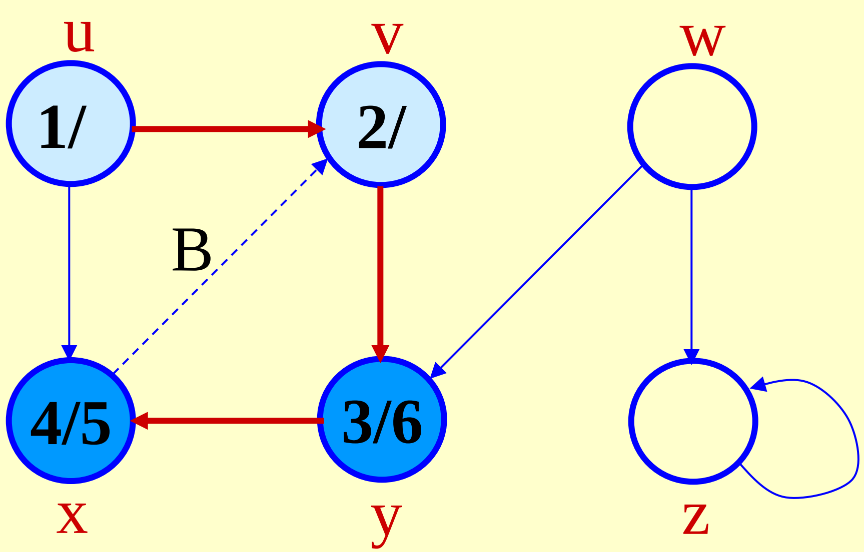
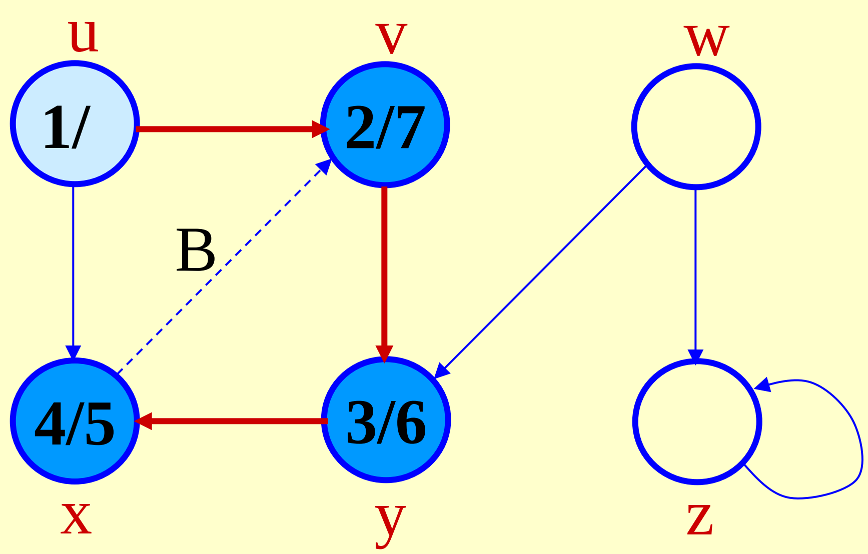


Example (DFS)



Example (DFS)



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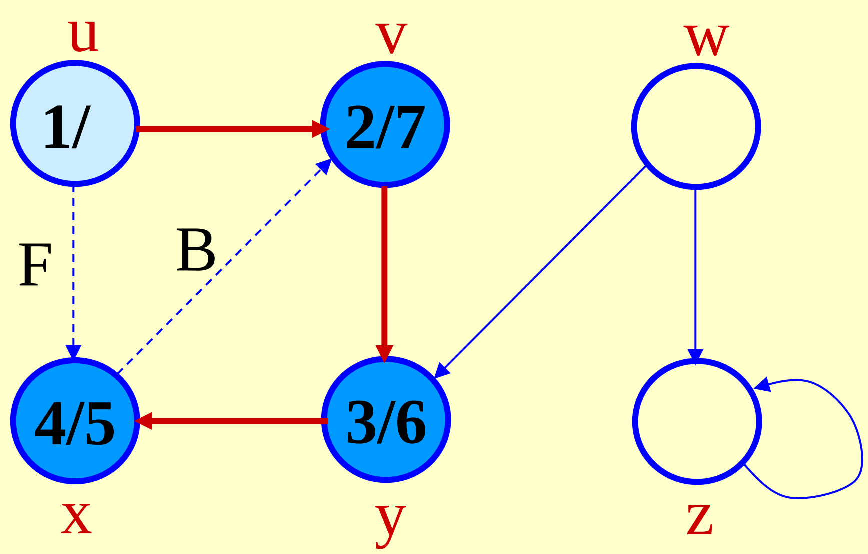

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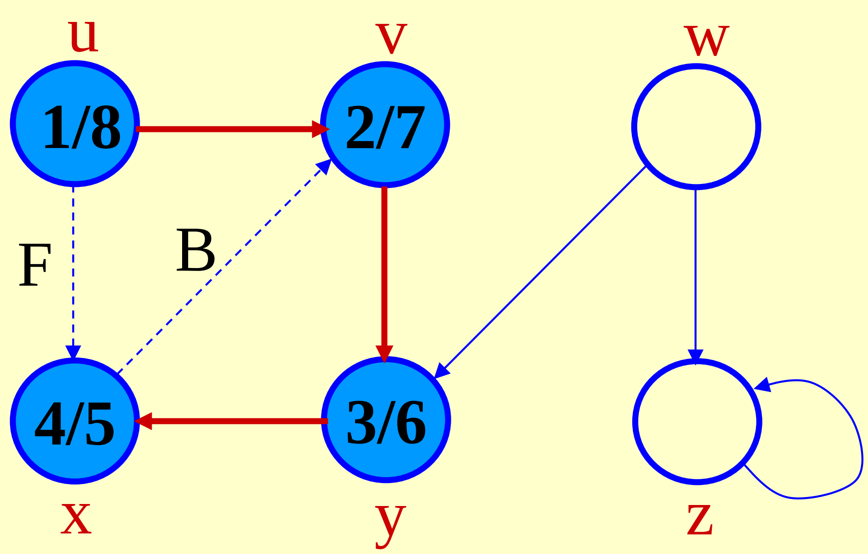

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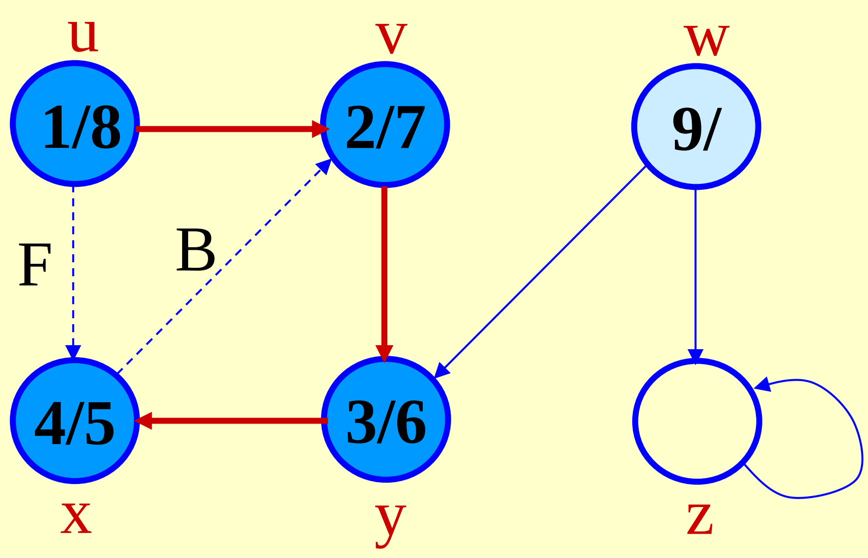

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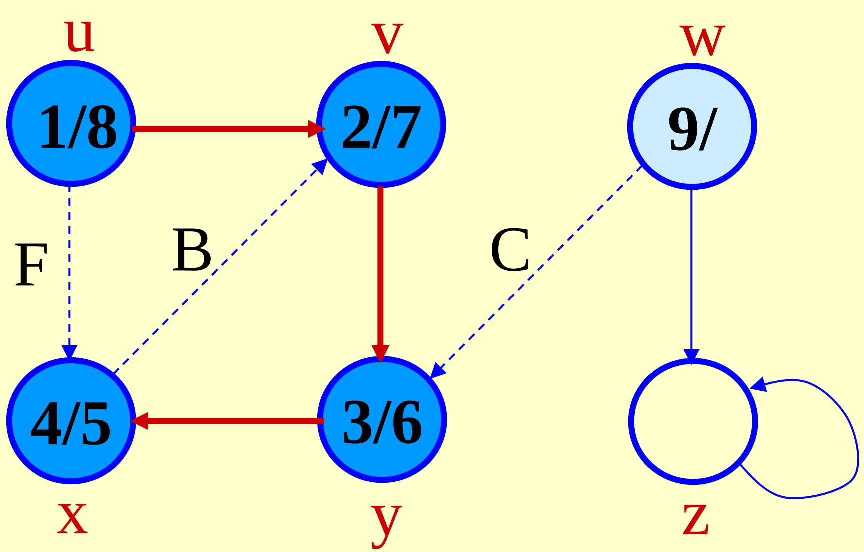

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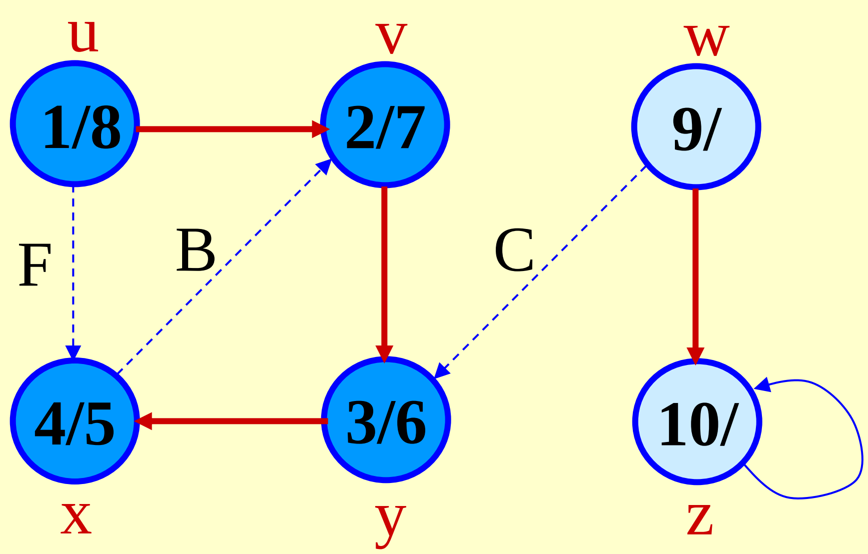

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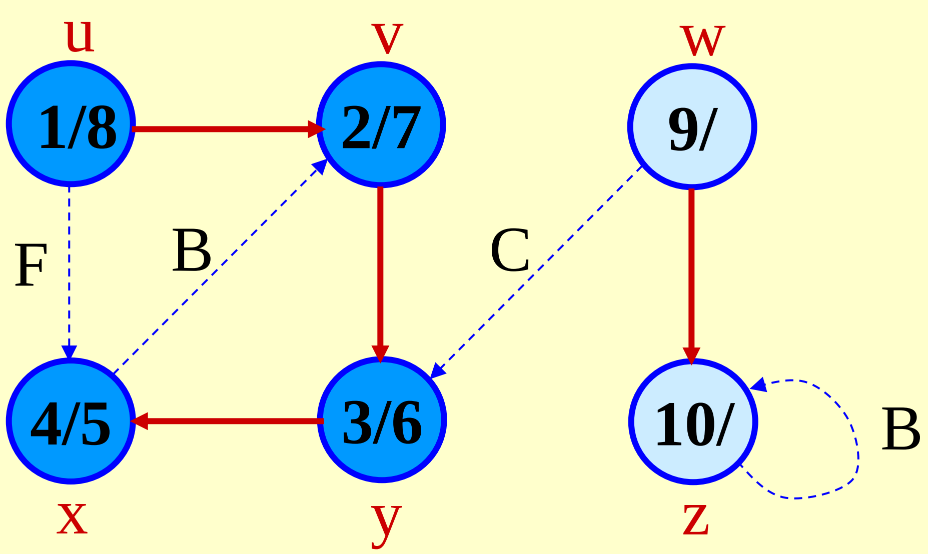

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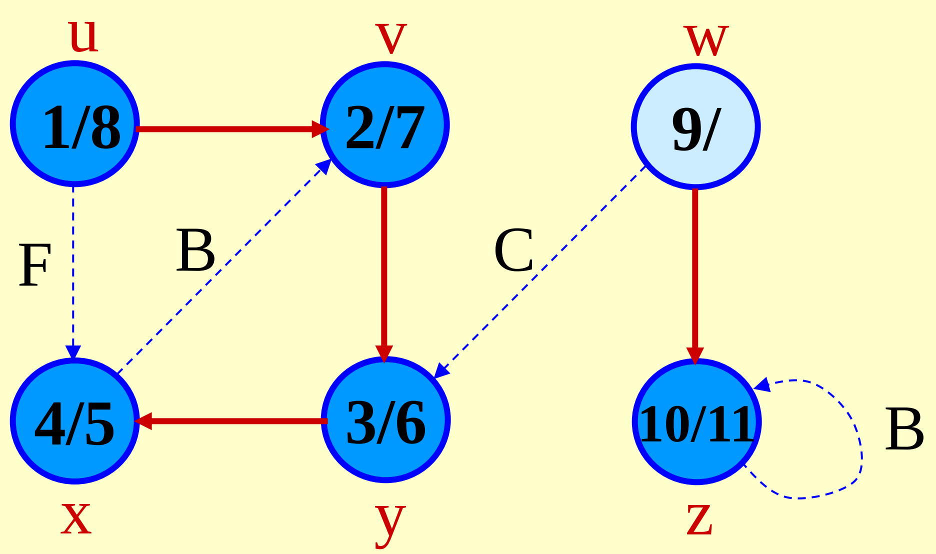

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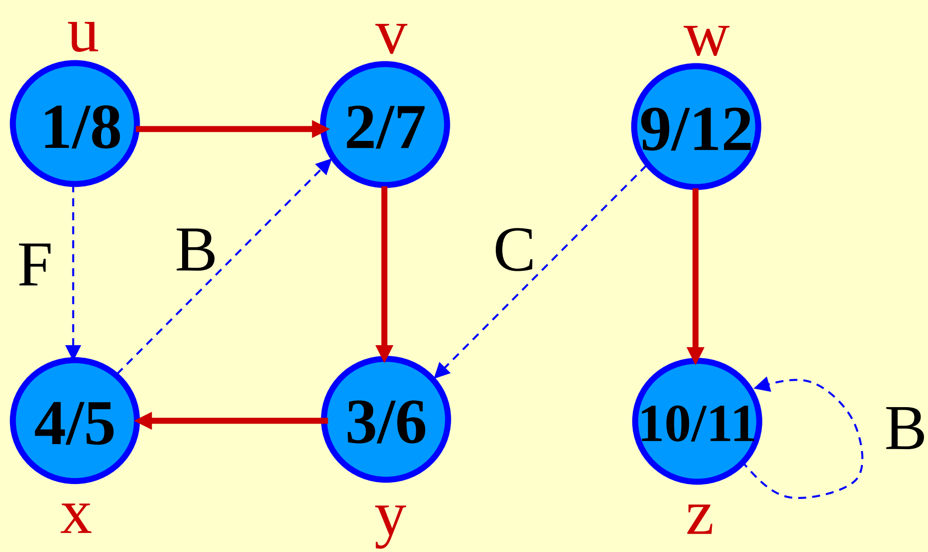


Analysis of DFS

- Loops on lines 1-2 & 5-7 take $\Theta(V)$ time, excluding time to execute DFS-Visit.

- DFS-Visit is called once for each white vertex $v \in V$ when it's painted gray the first time. Lines 3-6 of DFS-Visit is executed $|\mathrm{Adj}[v]|$ times. The total cost of executing DFS-Visit is $\sum_{v \in V} |\mathrm{Adj}[v]| = \Theta(E)$

- Total running time of DFS is $\Theta(|V| + |E|)$.

Parenthesis Theorem

Theorem 22.7

For all *u*, *v*, exactly one of the following holds:

1. $d[u] < f[u] < d[v] < f[v]$ or $d[v] < f[v] < d[u] < f[u]$ and neither *u* nor *v* is a descendant of the other.
2. $d[u] < d[v] < f[v] < f[u]$ and *v* is a descendant of *u*.
3. $d[v] < d[u] < f[u] < f[v]$ and *u* is a descendant of *v*.

- So $d[u] < d[v] < f[u] < f[v]$ *cannot* happen.
- Like parentheses:
  - OK: ( ) [ ] ([ ]) [( )]
  - Not OK: ([ )] [( ])

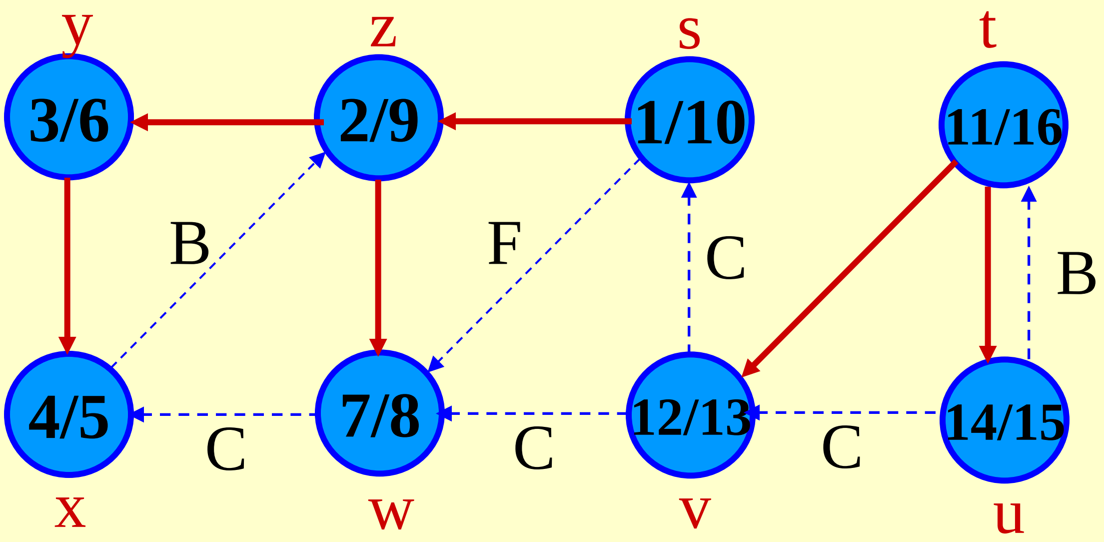
$\underset{d[v]}{(}\quad \underset{f[v]}{)}$

$\underset{d[v]}{(}\quad \overset{d[u]}{[}\quad \underset{f[v]}{)}\quad \overset{f[u]}{]}$

Corollary

v is a proper descendant of *u* if and only if $d[u] < d[v] < f[v] < f[u]$.

Example (Parenthesis Theorem)



(s (z (y (x x) y) (w w) z) s) (t (v v) (u u) t)

Depth-First Trees

- Predecessor subgraph defined slightly different from that of BFS.
- The predecessor subgraph of DFS is $G_\pi = (V, E_\pi)$ where $E_\pi = \{(\pi[v], v) : v \in V \text{ and } \pi[v] \neq nil\}$.
  - » How does it differ from that of BFS?
  - » The predecessor subgraph G_π forms a *depth-first forest* composed of several *depth-first trees*. The edges in E_π are called *tree edges*.

Definition:

Forest: An acyclic graph G that may be disconnected.

White-path Theorem

Theorem 22.9

v is a descendant of u in *DF-tree* if and only if at time $d[u]$, there is a path $u \leadsto v$ consisting of only white vertices. (Except for u, which was *just* colored gray.)

Classification of Edges

- **Tree edge:** in the depth-first forest. Found by exploring (u, v).
- **Back edge:** (u, v), where u is a descendant of v (in the depth-first tree).
- **Forward edge:** (u, v), where v is a descendant of u, but not a tree edge.
- **Cross edge:** any other edge. Can go between vertices in same depth-first tree or in different depth-first trees.

Theorem:
In DFS of an undirected graph, we get only tree and back edges. No forward or cross edges.