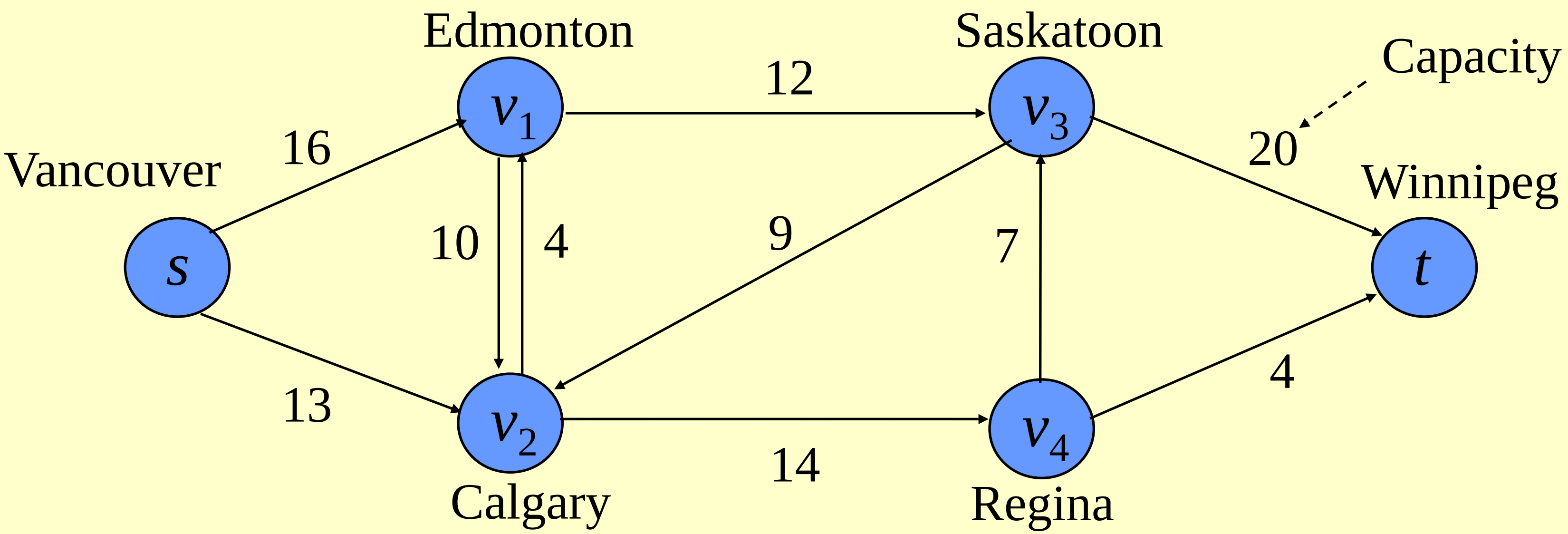
Network Flow

- What is a network?
- Flow network and flows
- Ford-Fulkerson method
  - Residual networks
  - Augmenting paths
  - Cuts of flow networks
- Max-flow min-cut theorem

Chapter 26: Maximum Flow

- **A directed graph is interpreted as a flow network:**
  - **A material coursing through a system from a source, where the material is produced, to a sink, where it is consumed.**
  - **The source produces the material at some steady rate, and the sink consumes the material at the same rate.**
- **Maximum problem: to compute the greatest rate at which material can be shipped from the source to the sink.**

- Example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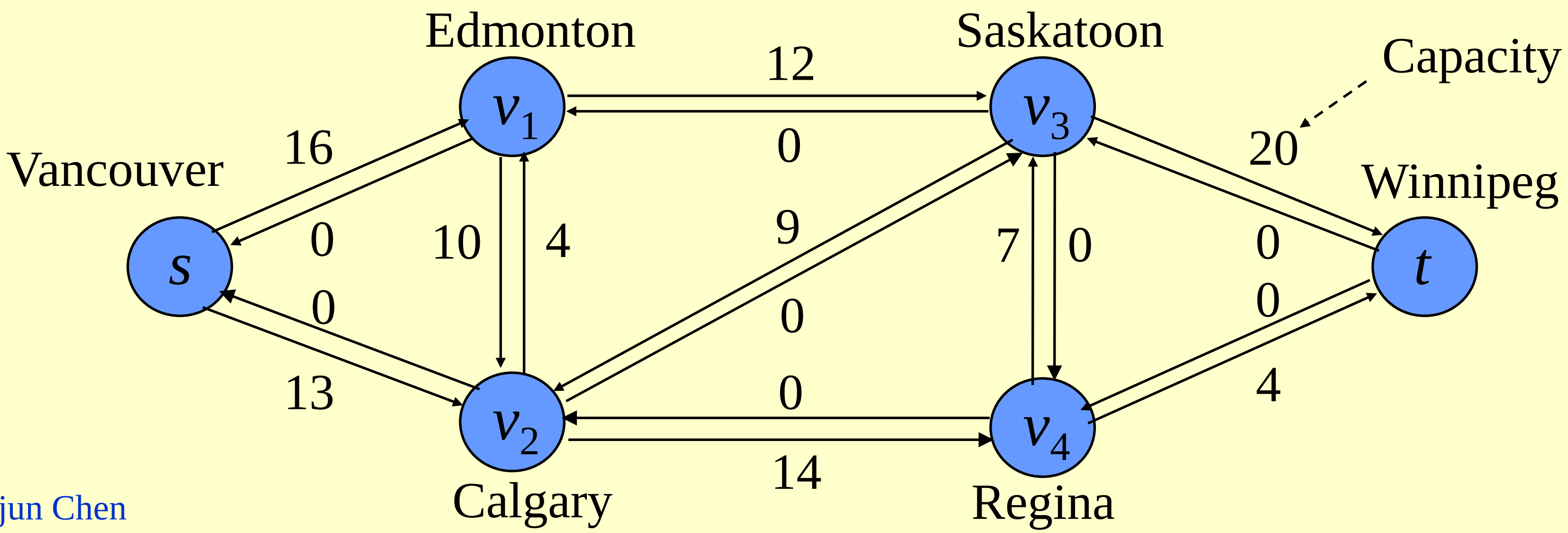
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- Applications which can be modeled by the maximum flow
  - Liquids flowing through pipes
  - Parts through assembly lines
  - current through electrical network
  - information through communication network

- Definition – flow networks and flows
  - A flow network $G = (V, E)$ is a directed graph in which each edges $(u, v) \in E$ has a nonnegative capacity $c(u, v) \geq 0$.
  - source: s; sink: t
  - For every vertex $v \in V$, there is a path:

    $s \rightsquigarrow v \rightsquigarrow t$

  - A flow in G is a real-valued function f: $V \times V \rightarrow \mathbf{R}$ that satisfies the following properties:

    Capacity constraint: For all $u, v \in V$, $f(u, v) \leq c(u, v)$.

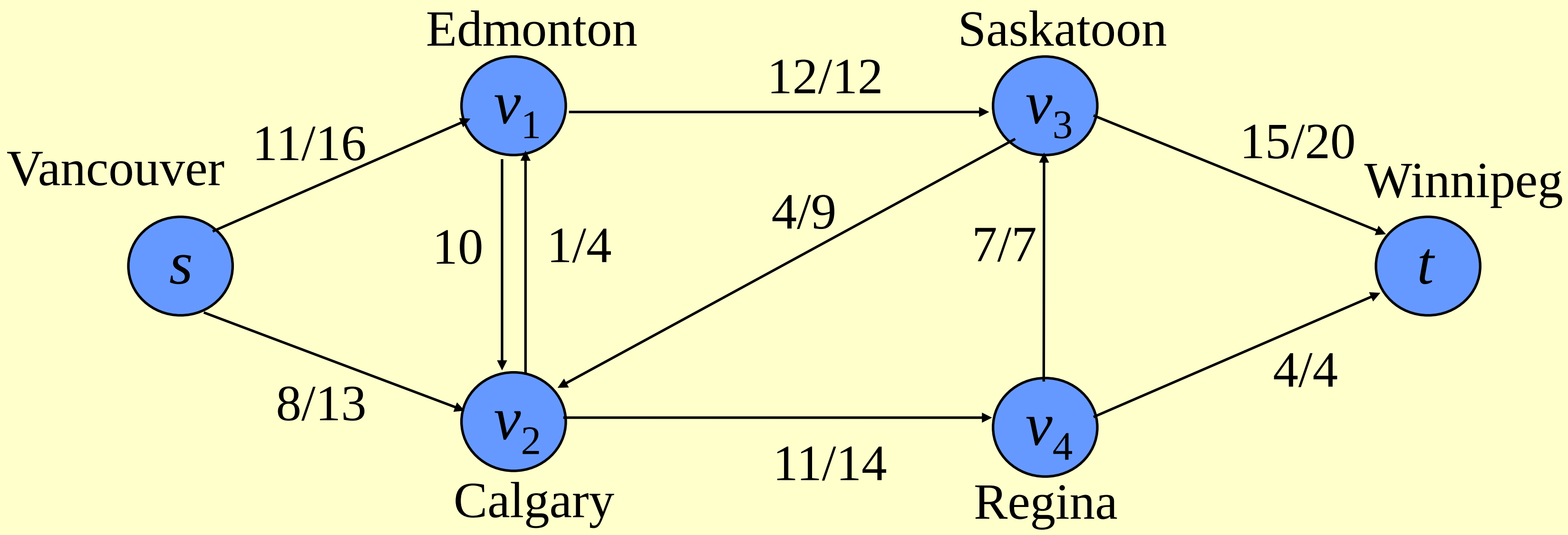
    Skew symmetry: For all $u, v \in V$, $f(u, v) = - f(v, u)$.

    Flow conservation: For all $u \in V - \{s, t\}$, $\sum_{v \in V} f(u,v) = 0$.

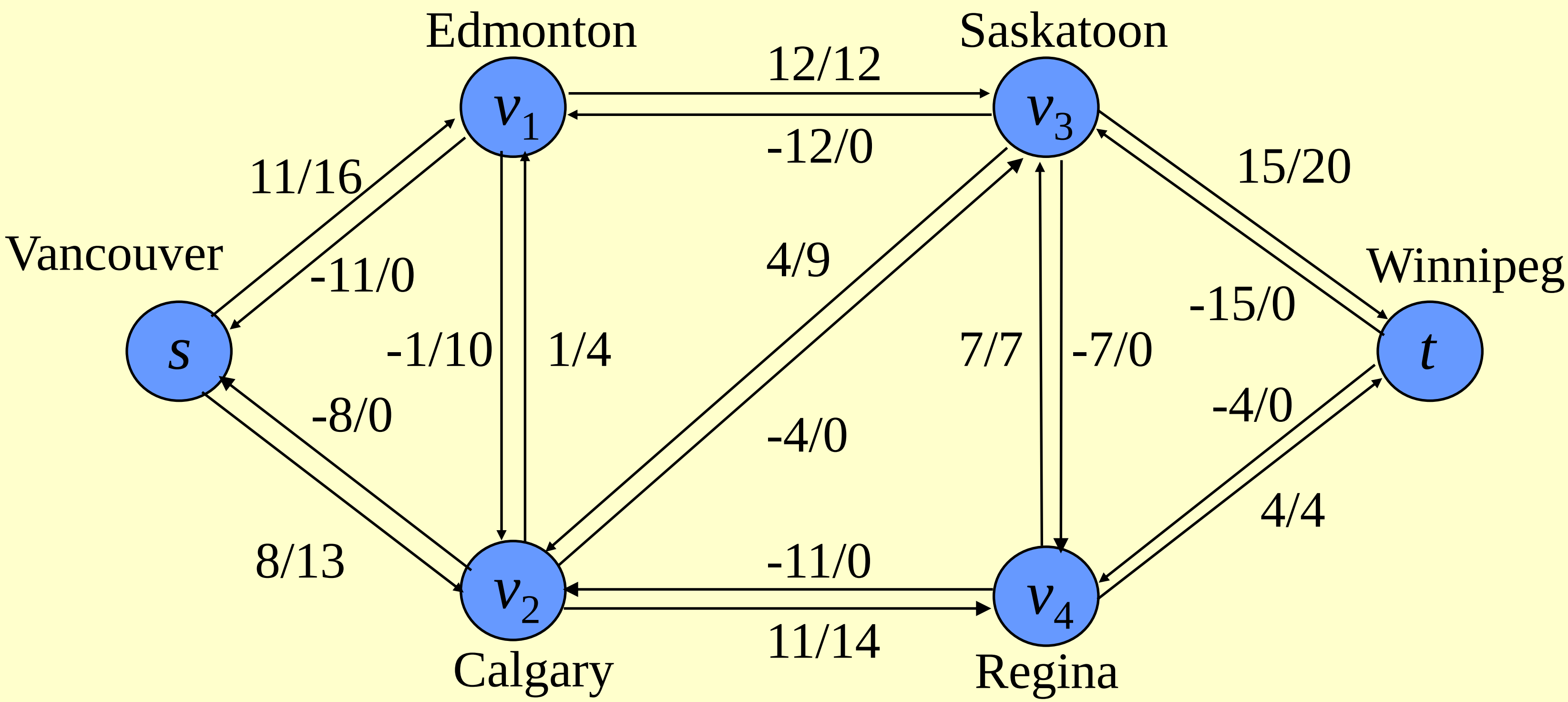
The quantity *f(u, v)*, which can be positive, zero, or negative, is called the ***flow*** from vertex *u* to vertex *v*. The value of a flow *f* is defined as the total flow out of the source

$$|f| = \sum_{v \in V} f(s, v)$$

- Example



■ Example



$\sum_{v \in V} f(u, v) = 0$. The total flow out of a vertex is 0.

$\sum_{u \in V} f(u, v) = 0$. The total flow into a vertex is 0.

The *total positive flow* entering a vertex v is defined by

$$\sum_{u \in V, f(u,v)>0} f(u,v)$$

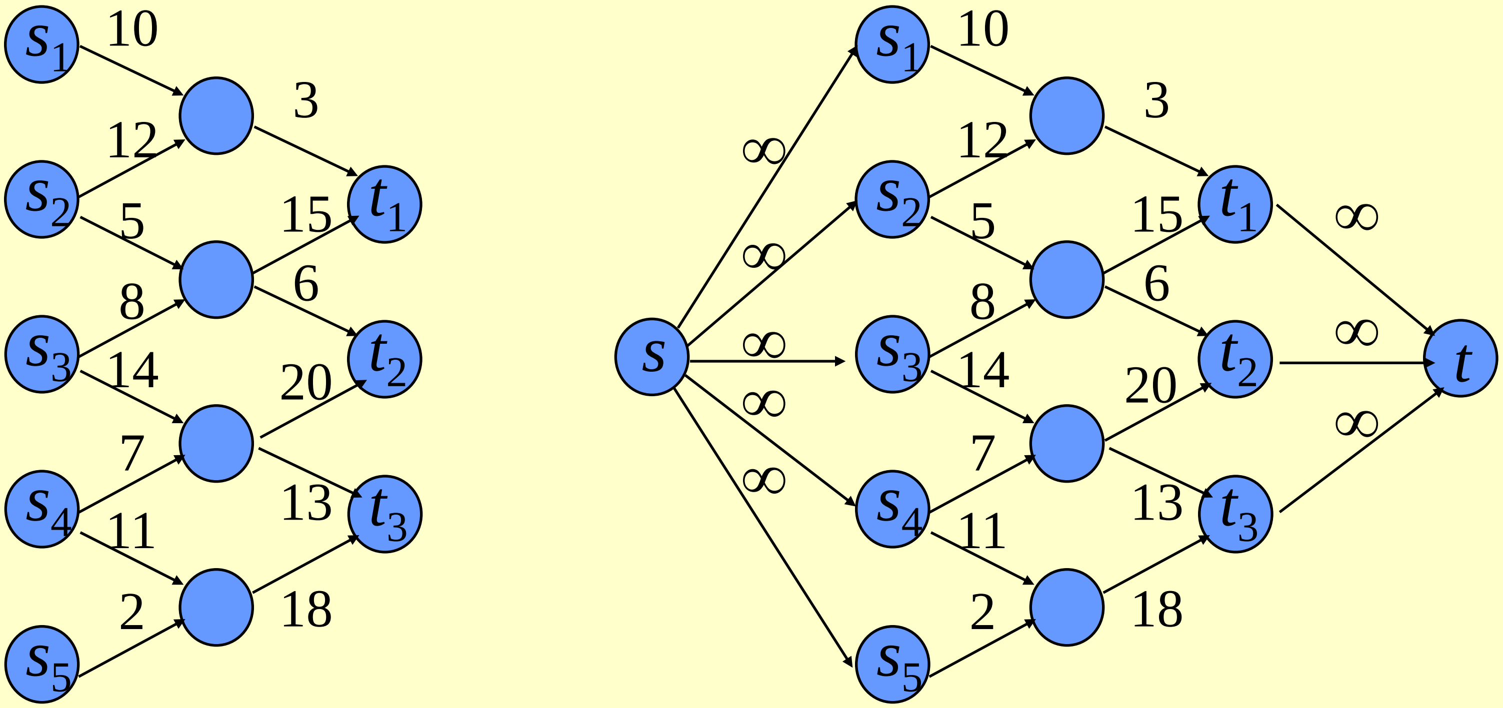
The *total net flow* at a vertex is the total positive flow leaving the vertex minus the total positive flow entering the vertex.

The *interpretation* of the flow-conservation property:

- The total positive flowing entering a vertex other than the source or sink must equal the total positive flow leaving that vertex.
- For all $u \in V - \{s, t\}$, $\sum_{v \in V} f(u,v) = 0$. That is, the total flow out of u is 0.
  For all $v \in V - \{s, t\}$, $\sum_{u \in V} f(u,v) = 0$. That is, the total flow into v is 0.

■Networks with multiple sources and sinks

- Introduce *supersource* *s* and *supersink* *t*



- Working with flows
  - implicit summation notation

    $f(X, Y) = \sum_{x \in X} \sum_{y \in Y} f(x, y)$

    The flow-conservation constraint can be re-expressed as

    $f(u, V) = 0$ for all $u \in V - \{s, t\}$.

  - **Lemma 26.1** Let $G = (V, E)$ be a flow network, and let f be a flow in G. Then, the following equalities hold:

    1. For all $X \subseteq V$, we have $f(X, X) = 0$.
    2. For all $X, Y \subseteq V$, we have $f(X, Y) = - f(Y, X)$.
    3. For all $X, Y, Z \subseteq V$ with $X \cap Y = ø$, we have the sums

    $$f(X \cup Y, Z) = f(X, Z) + f(Y, Z),$$

    $$f(Z, X \cup Y) = f(Z, X) + f(Z, Y).$$

- Working with flows
  - $|f| = f(V, t)$

    $|f| = f(s, V)$

    $= f(V, V) – f(V – s, V)$

    $= - f(V – s, V)$

    $= f(V, V – s)$

    $= f(V, t) + f(V, V – s – t)$

    $= f(V, t)$

- The Ford-Fulkerson method
  - *The maximum-flow problem*: given a flow network *G* with source *s* and sink *t*, we wish to find a flow of maximum value.
  - important concepts:

    residual networks

    augmenting paths

    cuts

Ford-Fulkerson-Method(*G*, *s*, *t*)

1. Initialize flow *f* to 0
2. **while** there exists an augmenting path *p*
3. **do** augment flow *f* along *p*
4. **return** *f*

- Residual networks
  - Given a flow network and a flow, the residual network consists of edges that can admit more flow.
  - Let f be a flow in $G = (V, E)$ with source s and sink t. Consider a pair of vertices $u, v \in V$. The amount of *additional* flow we can push from u to v before exceeding the capacity $c(u, v)$ is the ***residual capacity*** of (u, v), given by

$$c_f(u, v) = c(u, v) - f(u, v).$$

  - Example

    If $c(u, v) = 16$ and $f(u, v) = 11$, then $c_f(u, v) = 16 - 11 = 5$.

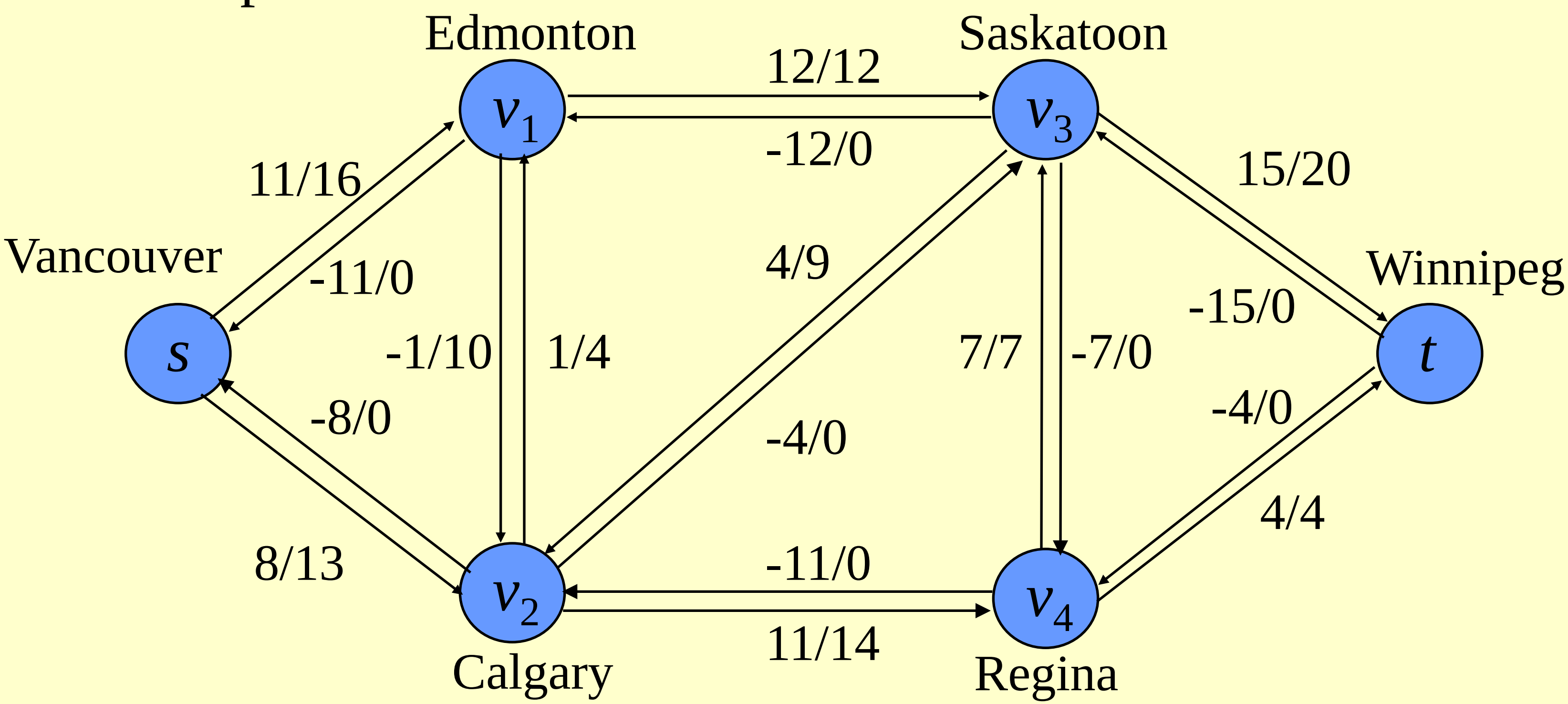
    If $c(u, v) = 16$ and $f(u, v) = -4$, then $c_f(u, v) = 16 - (-4) = 20$.

Residual networks

- Given a flow network $G = (V, E)$ and a flow f, the ***residual network*** of G induced by f is $G_f = (V, E_f)$, where

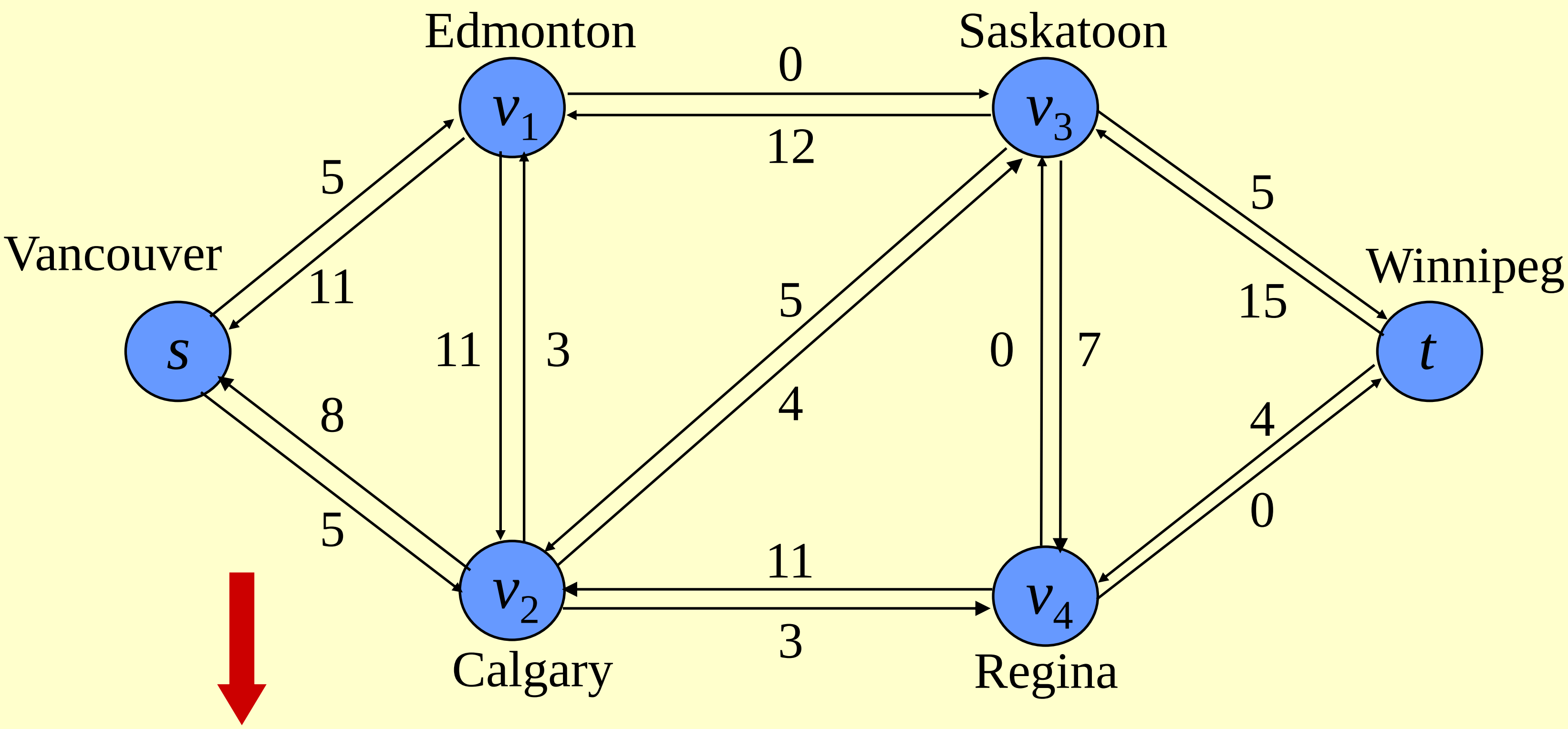
  $E_f = \{(u, v) \in V \times V: c_f(u, v) > 0\}$.

- Example



■ Residual networks

residual network:



$|E_f| \leq 2|E|$

Residual networks

Lemma 26.2 Let $G = (V, E)$ be a network with source s and sink t, and let f be a flow in G. Let G_f be the residual network of G induced by f, and let f' be a flow in G_f. Then, the flow sum $f + f'$ (defined by $(f + f')(u, v) = f(u, v) + f'(u, v)$) is a flow in G with value $|f + f'| = |f| + |f'|$.

Proof. We must verify that skew symmetry, the capacity constraints, and flow conservation are obeyed.

skew symmetry:

$(f + f')(u, v) = f(u, v) + f'(u, v) = - f(v, u) - f'(v, u)$

$= - (f(v, u) + f'(v, u)) = - (f + f')(v, u).$

capacity constraint:

$$(f + f')(u, v) = f(u, v) + f'(u, v)$$
$$\leq f(u, v) + (c(u, v) - f(u, v))$$
$$= c(u, v).$$

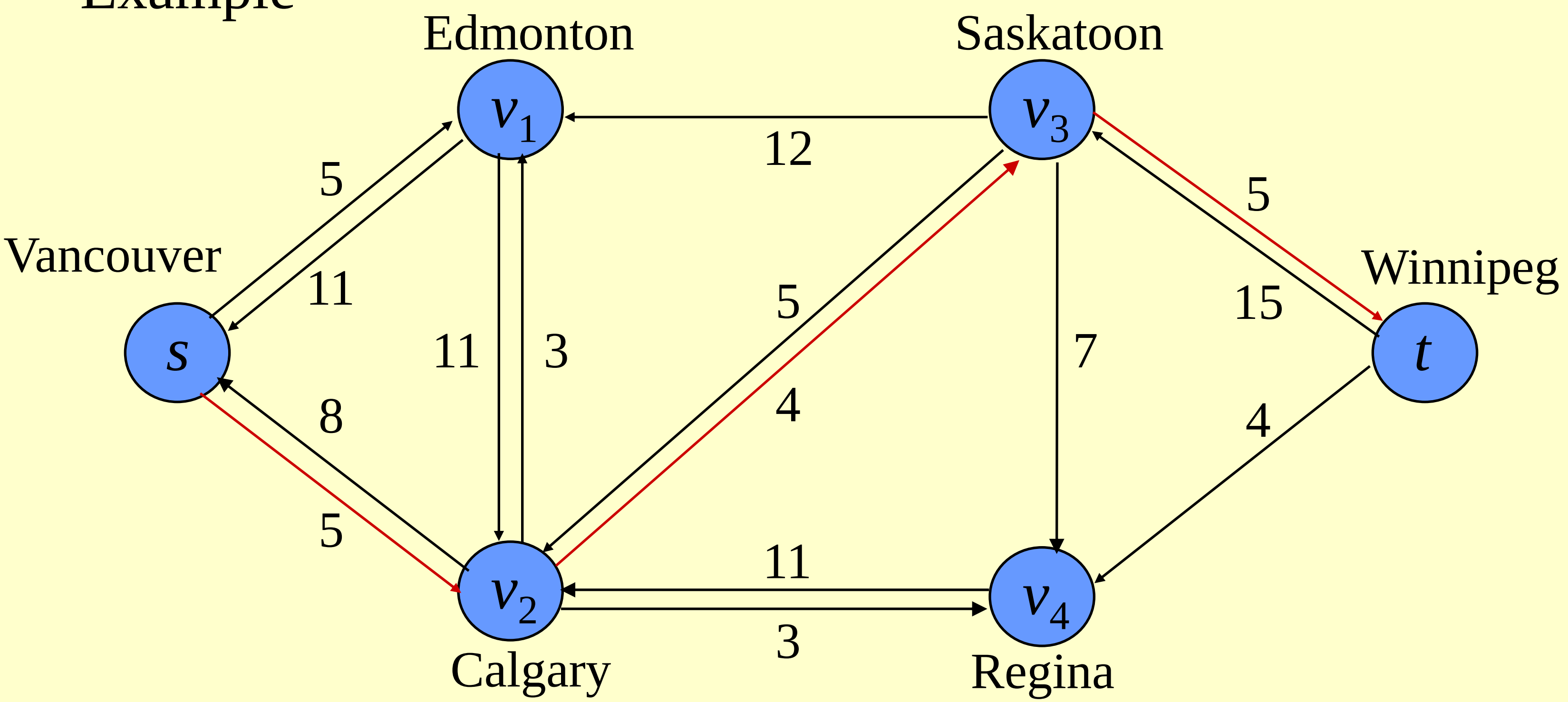
flow conservation:

$$\sum_{v \in V} (f + f')(u,v) = \sum_{v \in V} (f(u,v) + f'(u,v))$$
$$= \sum_{v \in V} f(u,v) + \sum_{v \in V} f'(u,v)$$
$$= 0 + 0 = 0.$$

Finally, we have

$$|f + f'| = \sum_{v \in V} (f + f')(s,v) = \sum_{v \in V} (f(s,v) + f'(s,v))$$
$$= \sum_{v \in V} f(s,v) + \sum_{v \in V} f'(s,v)$$
$$= |f| + |f'|$$

- Augmenting paths
  - Given a flow network $G = (V, E)$ and a flow f, an augmenting path p is a simple path from s to t in the residual network G_f.
  - Example



- Augmenting paths
  - In the above above residual network, path $s \to v_2 \to v_3 \to t$ is an augmenting path.
  - We can increase the flow through each edge of this path by up to 4 units without violating a capacity constraint since the smallest residual capacity on this path is $c_f(v_2, v_3) = 4$.
  - *residual capacity of an augmenting path*

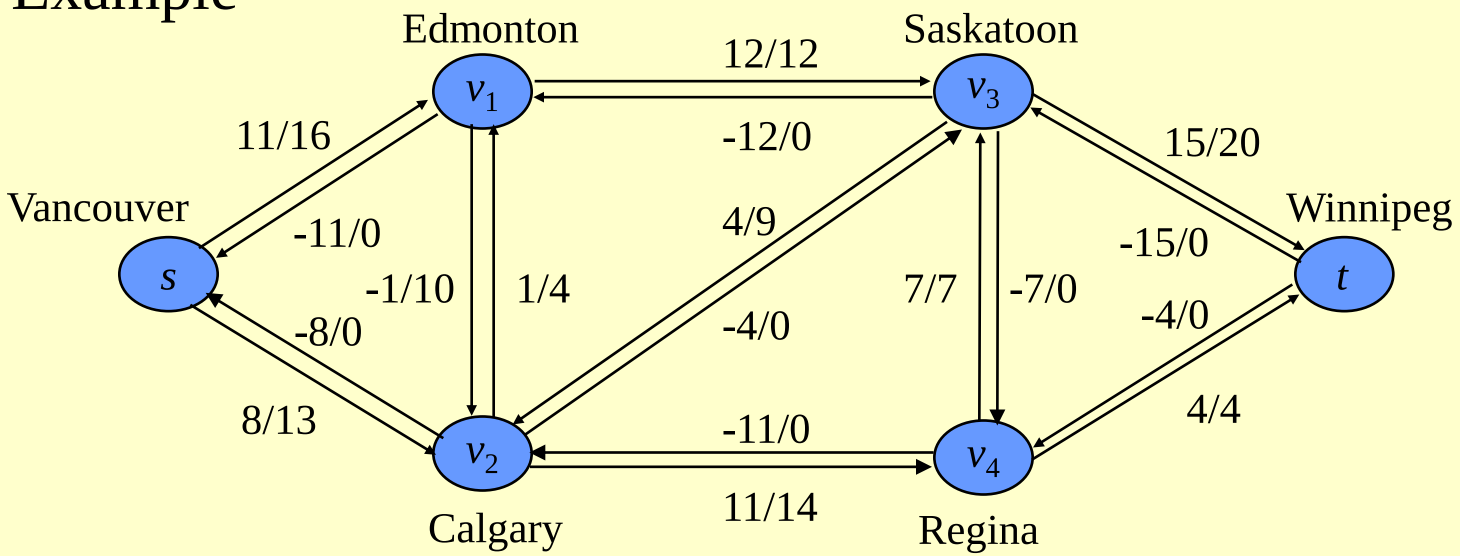
$$c_f(p) = \min\{c_f(u, v): (u, v) \text{ is on } p\}.$$

  - **Lemma 26.3** Let $G = (V, E)$ be a network, let f be a flow in G, and let p be an augmenting path in G_f. Define a function $f_p: V \times V \to \boldsymbol{R}$ by

$$f_p(u, v) = \begin{cases} c_f(p) & \text{if } (u, v) \text{ is on } p, \\ -\,c_f(p) & \text{if } (v, u) \text{ is on } p, \\ 0 & \text{otherwise.} \end{cases}$$

Then, f_p is a flow in G_f with value $|f_p| = c_f(p)$.

- Example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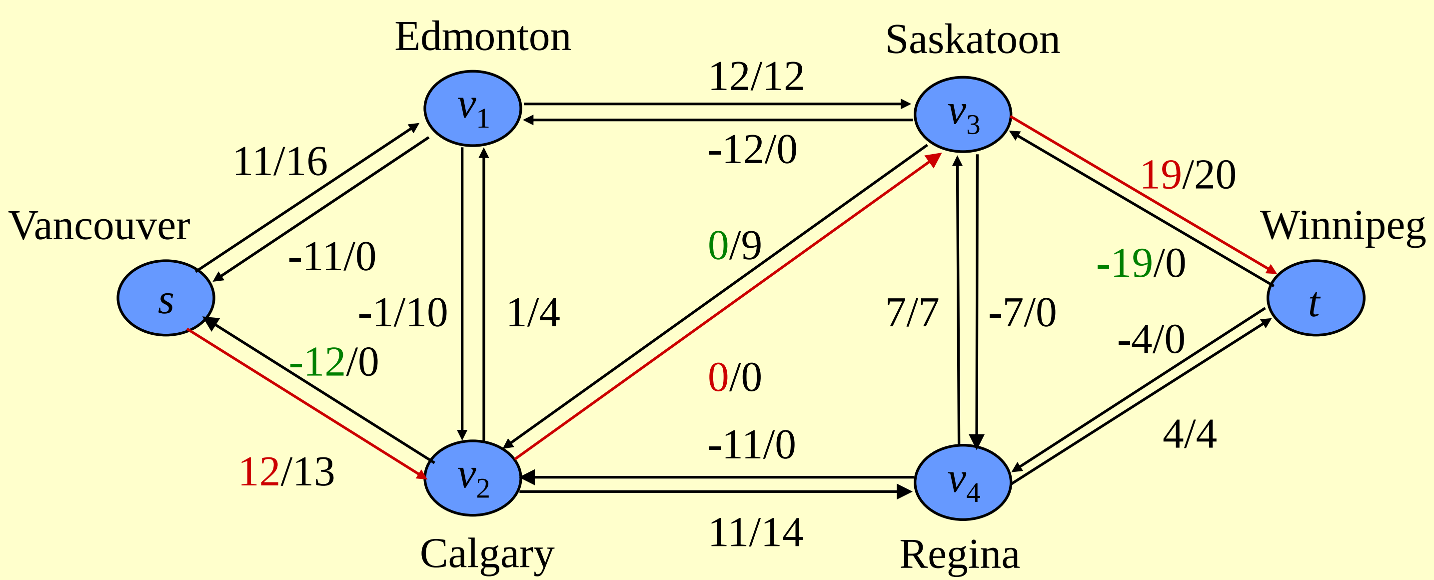
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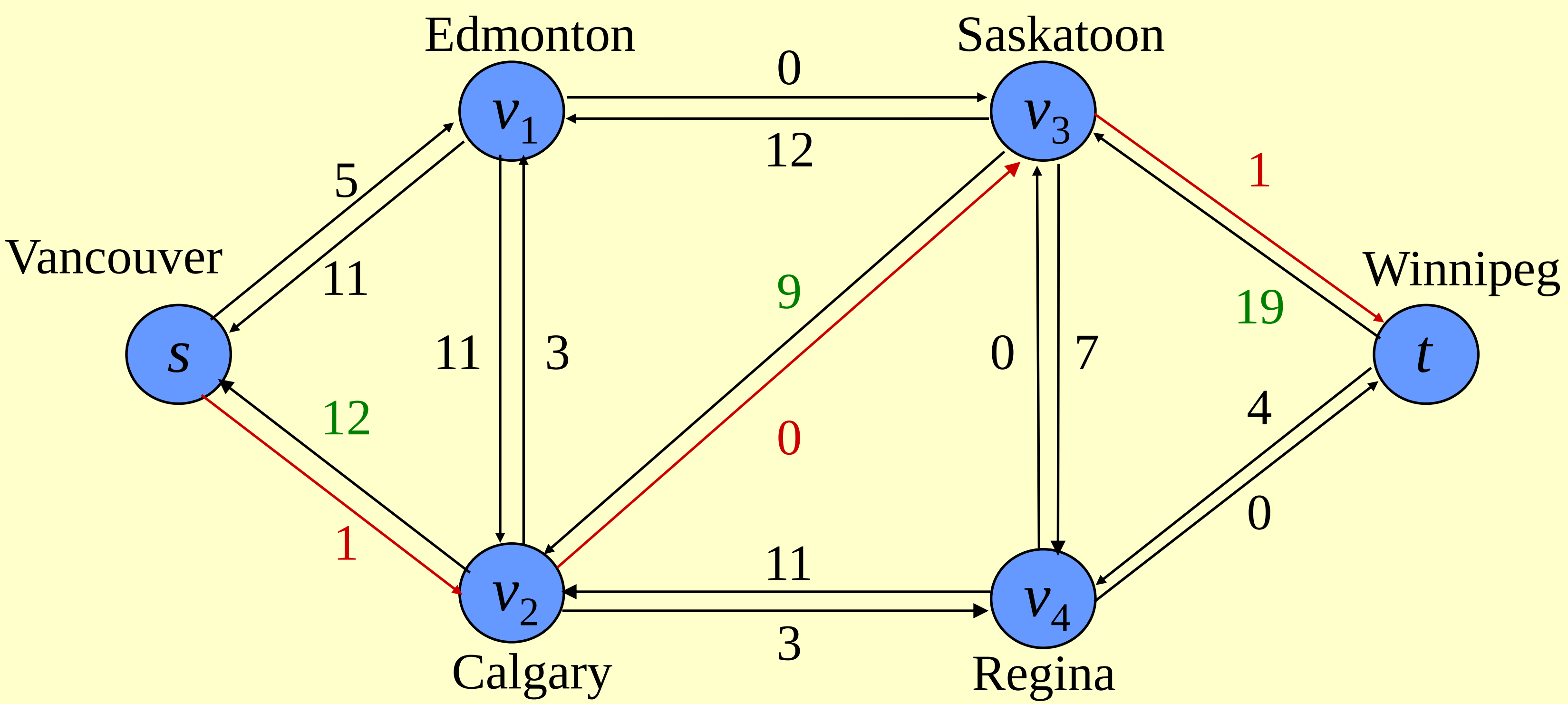
- Residual network induced by the new flow



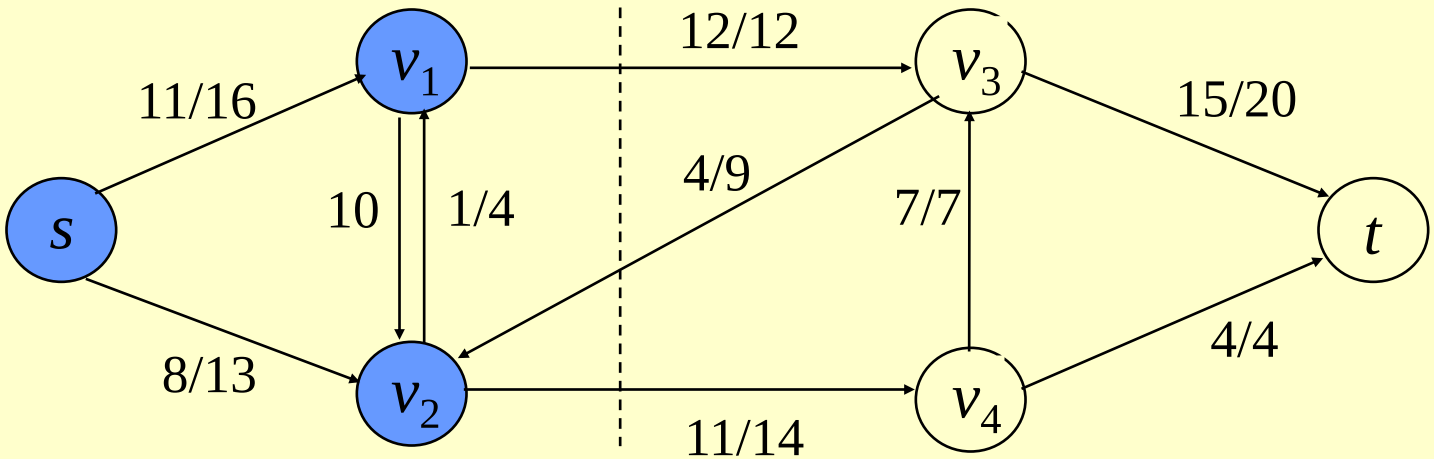
- Augmenting paths
  - **Corollary 26.4** Let $G = (V, E)$ be a network, let f be a flow in G, and let p be an augmenting path in G_f. Let f_p be defined as in Lemma 26.3. Define a function f': $V \times V \rightarrow \boldsymbol{R}$ by

    $$f' = f + f_p.$$

    Then, f' is a flow in G with value $|f'| = |f| + |f_p| > |f|$.

    Proof. Immediately from Lemma 26.2 and 26.3.
- Cuts of flow networks
  - The Ford-Fulkerson method repeatedly augments the flow along augmenting paths until a maximum flow has been found.
  - A flow is maximum if and only if its residual network contains no augmenting path.

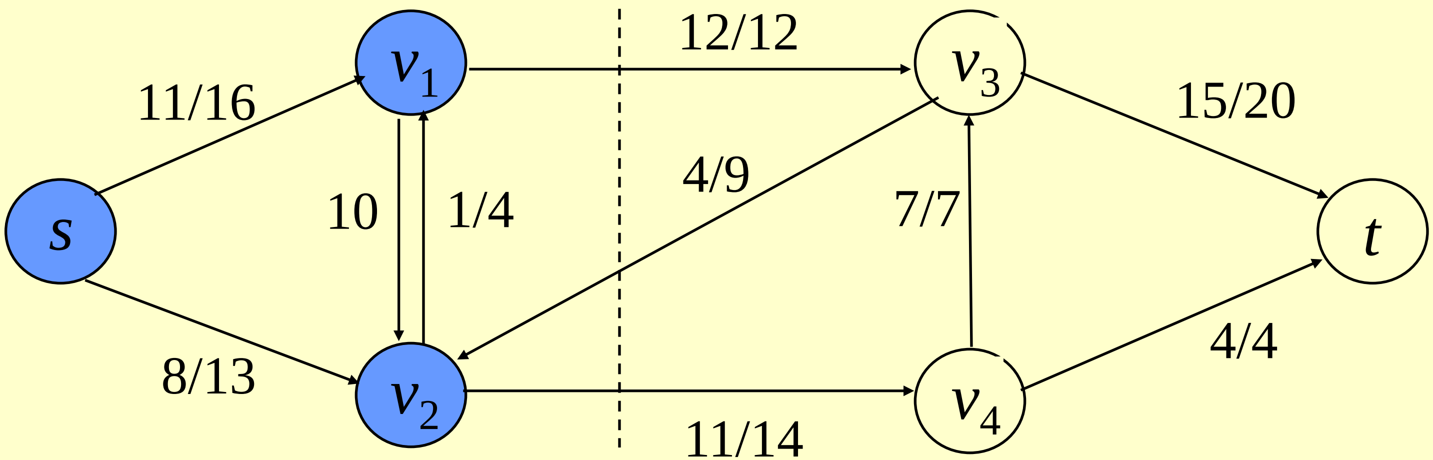
- Cuts of flow networks
  - A cut (S, T) of flow network $G = (V, E)$ is a partition of V into S and $T = V - S$ such that $s \in S$ and $t \in T$.
  - *net flow* across the cut (S, T) is defined to be $f(S, T)$.



$f(\{s, v_1, v_2\}, \{v_3, v_4, t\}) = f(v_1, v_3) + f(v_2, v_3) + f(v_2, v_4)$
$= 12 + (-4) + 11 = 19.$

The net flow across a cut (S, T) consists of positive flows in both direction.

- Cuts of flow networks
  - The capacity of the cut (*S*, *T*) is denoted by *c*(*S*, *T*), which is computed only from edges going from *S* to *T*.



$c(\{s, v_1, v_2\}, \{v_3, v_4, t\}) = c(v_1, v_3) + c(v_2, v_4)$
$= 12 + 14 = 26.$

- Cuts of flow networks

  - The following lemma shows that the net flow across any cut is the same, and it equals the value of the flow.

    Lemma 26.5 Let f be a flow is a flow network G with source s and sink t, and let (S, T) be a cut of G. Then, the net flow across (S, T) is $f(S, T) = |f|$.

    Proof. Note that $f(S - s, V) = 0$ by flow conservation. So we have

    $$\begin{aligned} f(S, T) &= f(S, V) - f(S, S) \\ &= f(S, V) \\ &= f(s, V) + f(S - s, V) \\ &= f(s, V) \\ &= |f|. \end{aligned}$$

- Cuts of flow networks
  - **Corollary 26.6** The value of any flow in a flow network G is bounded from above by the capacity of any cut of G.
    Proof.

$$
\begin{aligned}
|f| &= f(S, T) \\
&= \sum_{u \in S} \sum_{v \in T} f(u, v) \\
&\leq \sum_{u \in S} \sum_{v \in T} c(u, v) \\
&= c(S, T).
\end{aligned}
$$

- Max-flow min-cut theorem

Theorem 26.7 If f is a flow network $G = (V, E)$ with source s and sink t, then the following conditions are equivalent:

1. f is a maximum flow in G.

2. The residual network G_f contains no augmenting paths.

3. $|f| = c(S, T)$ for some cut (S, T) of G.

Proof. (1) $\Rightarrow$ (2): Suppose for the sake of contradiction that f is a maximum flow in G but that G_f has an augmenting path p. Then, by Corollary 26.4, the flow sum $f + f_p$, where f_p is given by Lemma 26.3, is a flow in G with value strictly greater than $|f|$, contradicting the assumption that f is a maximum flow.

- Max-flow min-cut theorem

Theorem 26.7 If f is a flow network $G = (V, E)$ with source s and sink t, then the following conditions are equivalent:

1. f is a maximum flow in G.

2. The residual network G_f contains no augmenting paths.

3. $|f| = c(S, T)$ for some cut (S, T) of G.

Proof. (2) $\Rightarrow$ (3): Suppose that G_f has no augmenting path. Define $S = \{v \in V$: there exists a path from s to v in $G_f\}$ and $T = V - S$. The partition (S, T) is a cut: we have $s \in S$ trivially and $t \notin S$ because there is no path from s to t in G_f. For each pair of vertices u and v such that $u \in S$ and $v \in T$, we have $f(u, v) = c(u, v)$, since otherwise $(u, v) \in E_f$, which would place v in set S. By Lemma 26.5, therefore, $|f| = f(S, T) = c(S, T)$.

- Max-flow min-cut theorem

Theorem 26.7 If f is a flow network $G = (V, E)$ with source s and sink t, then the following conditions are equivalent:

1. f is a maximum flow in G.

2. The residual network G_f contains no augmenting paths.

3. $|f| = c(S, T)$ for some cut (S, T) of G.

Proof. (3) $\Rightarrow$ (1): By Corollary 26.6, $|f| \leq c(S, T)$ for all cuts (S, T). The condition $|f| = c(S, T)$ thus implies that f is a maximum flow.

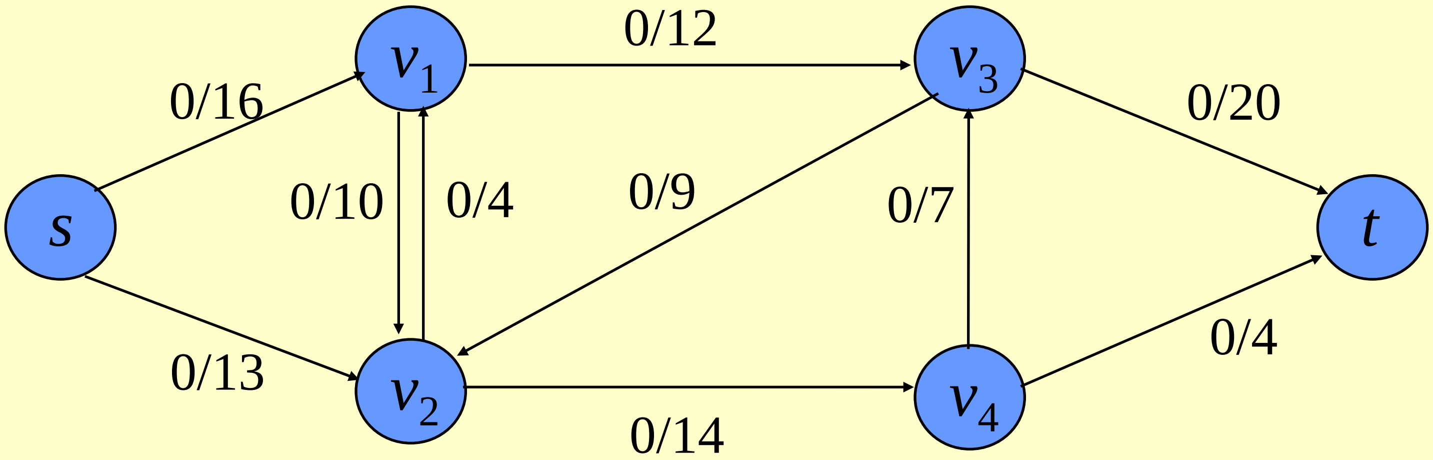
- Ford-Fulkerson algorithm

Ford_Fulkerson(G, s, t)

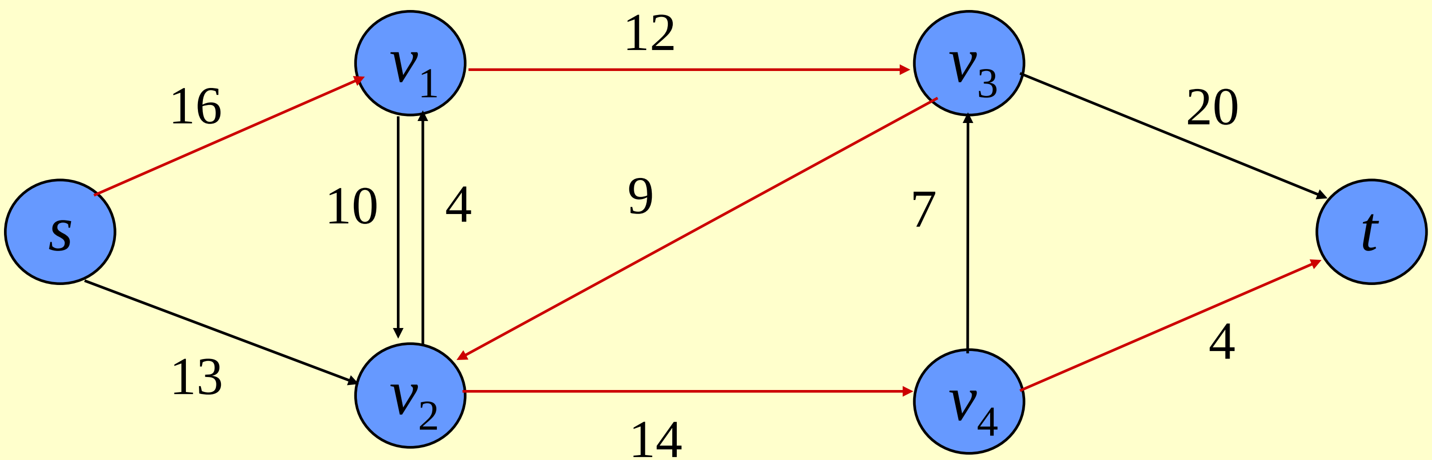
1. **for** each edge $(u, v) \in E(G)$
2. **do** $f(u, v) \leftarrow 0$
3. $f(v, u) \leftarrow 0$
4. **while** there exists a path p from s to t in G_f
5. **do** $c_f(p) \leftarrow \min\{c_f(u, v) : (u, v) \text{ is in } p\}$
6. **for** each edge (u, v) is in p
7. **do** $f(u, v) \leftarrow f(u, v) + c_f(p)$
8. $f(v, u) \leftarrow - f(u, v)$

- Sample trace

Initially, the flow on edge is 0.

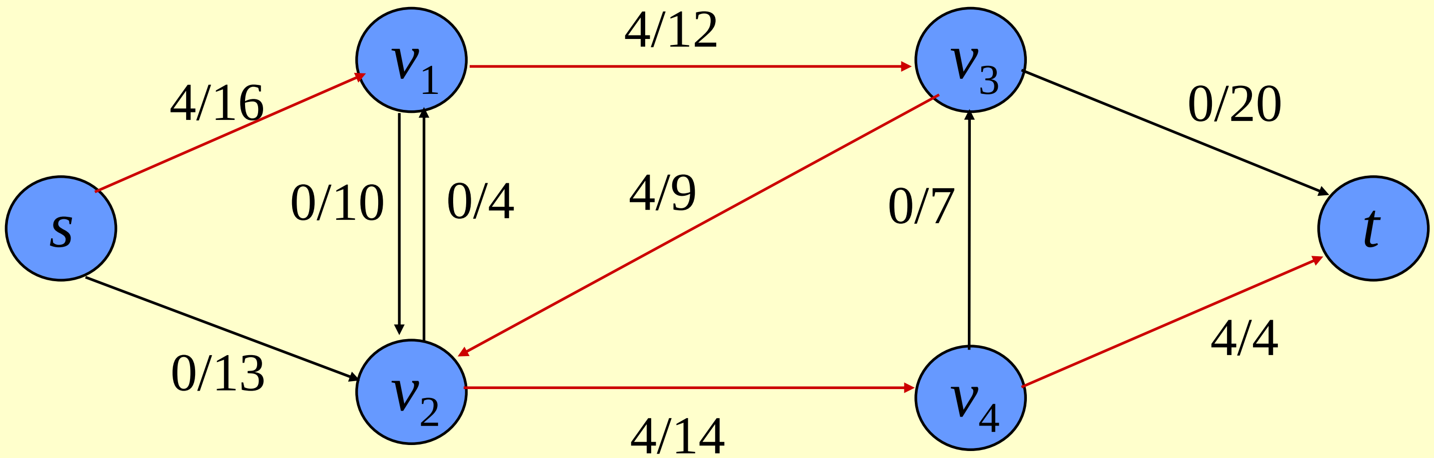


The corresponding residual network:



- Sample trace

Pushing a flow 4 on *p*1 (an augmenting path)



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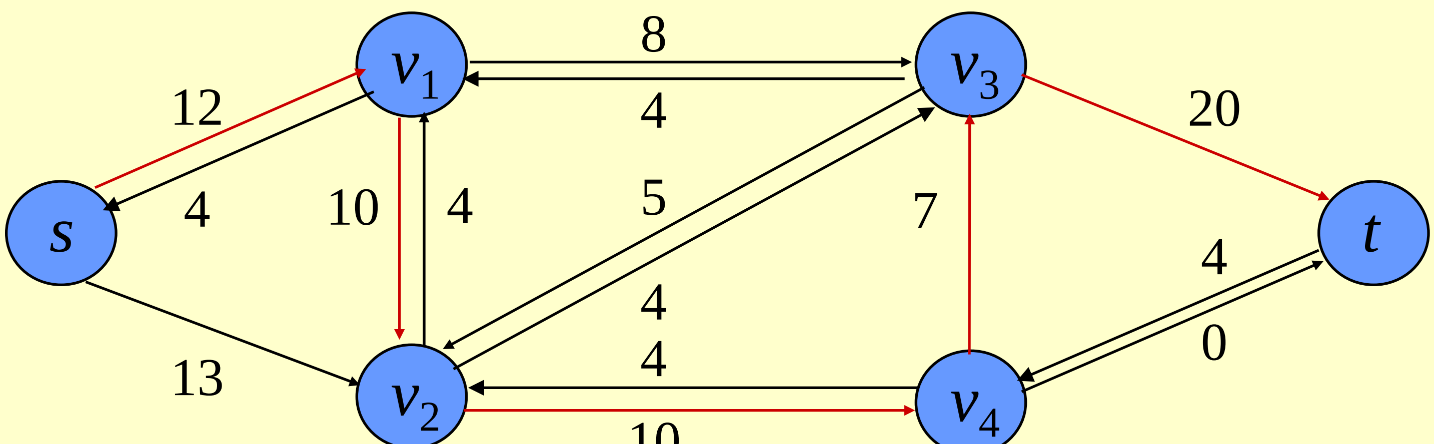
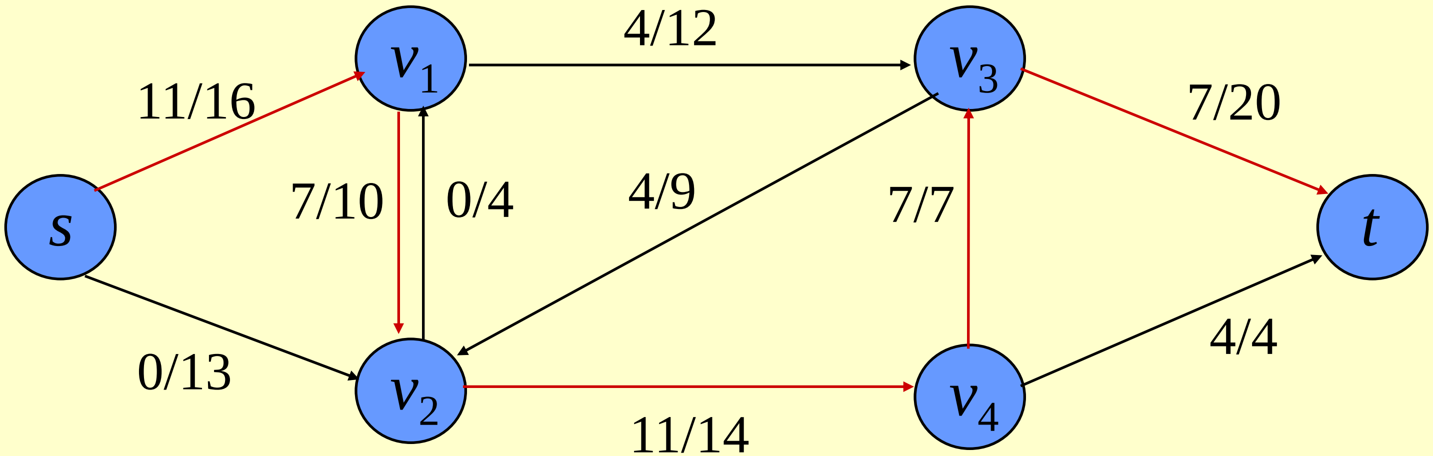
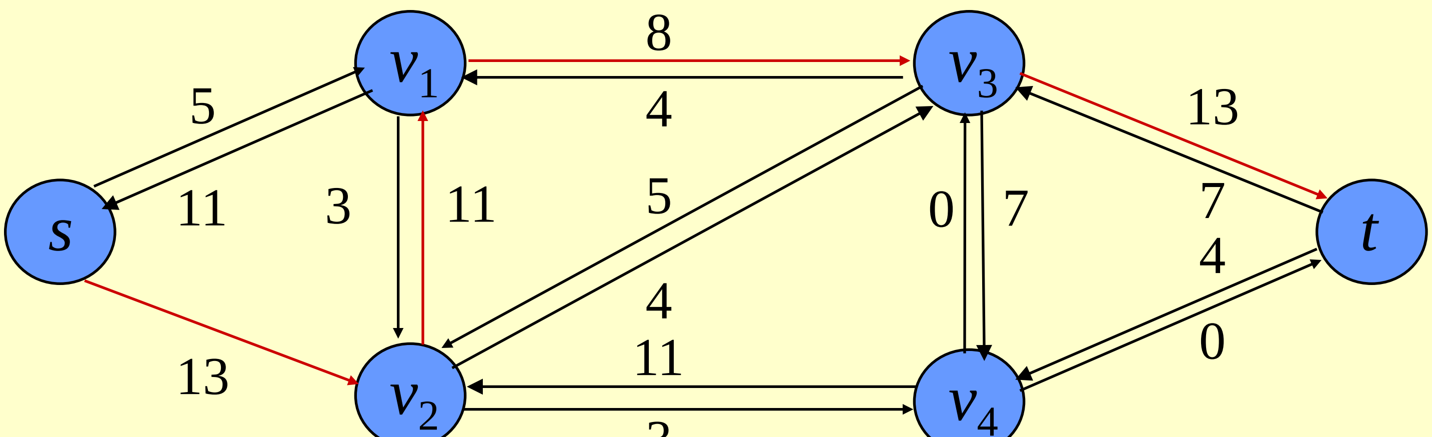


- Sample trace

Pushing a flow 7 on *p*2 (an augmenting path)

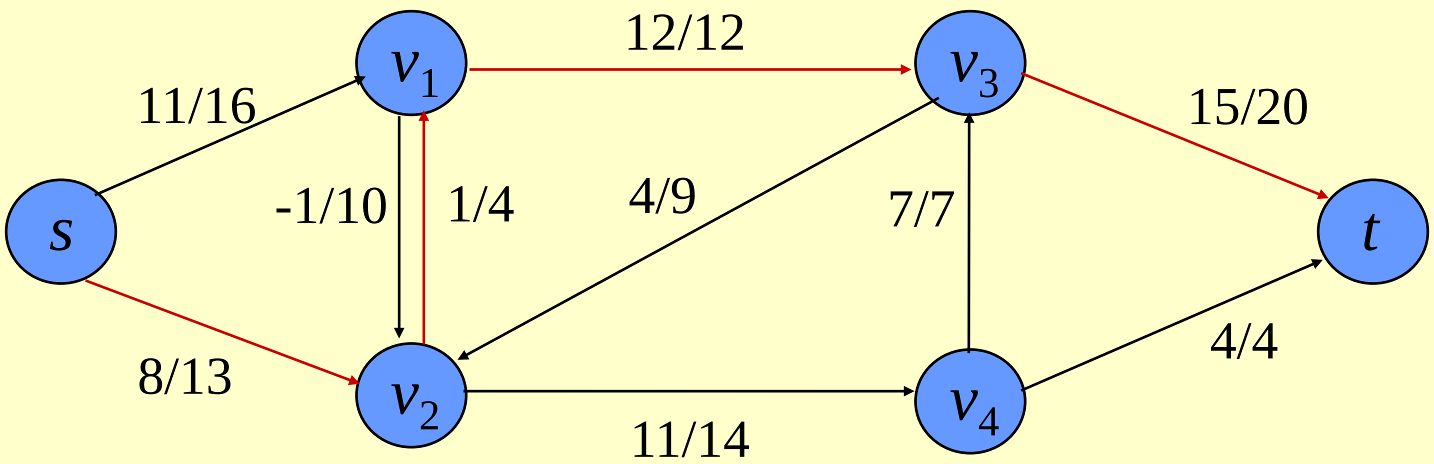


The corresponding residual network:

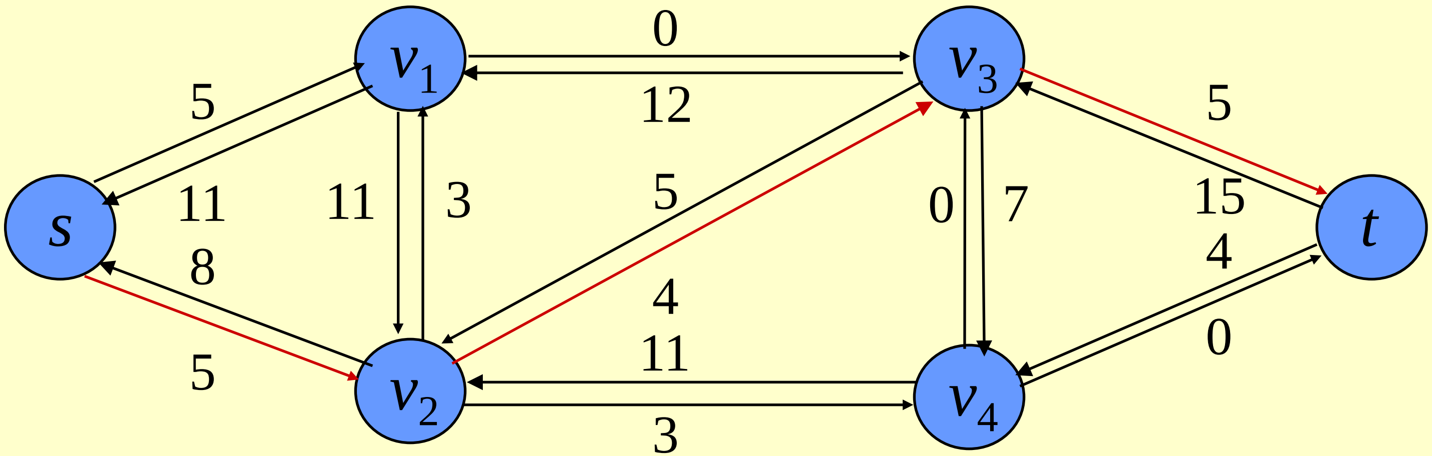


- Sample trace

Pushing a flow 8 on *p*3 (an augmenting path)

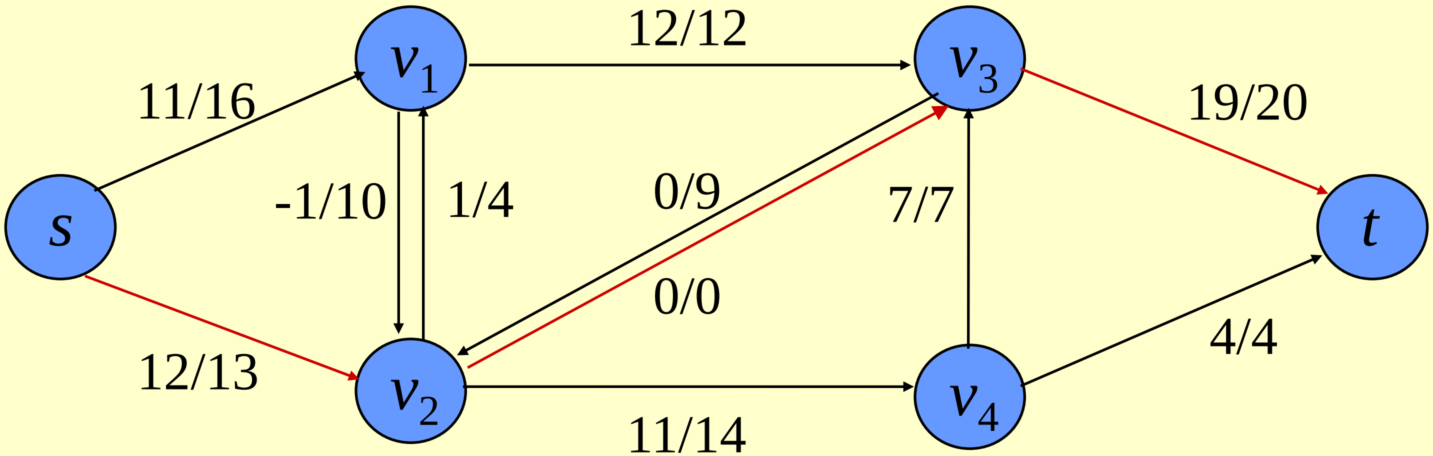


The corresponding residual network:

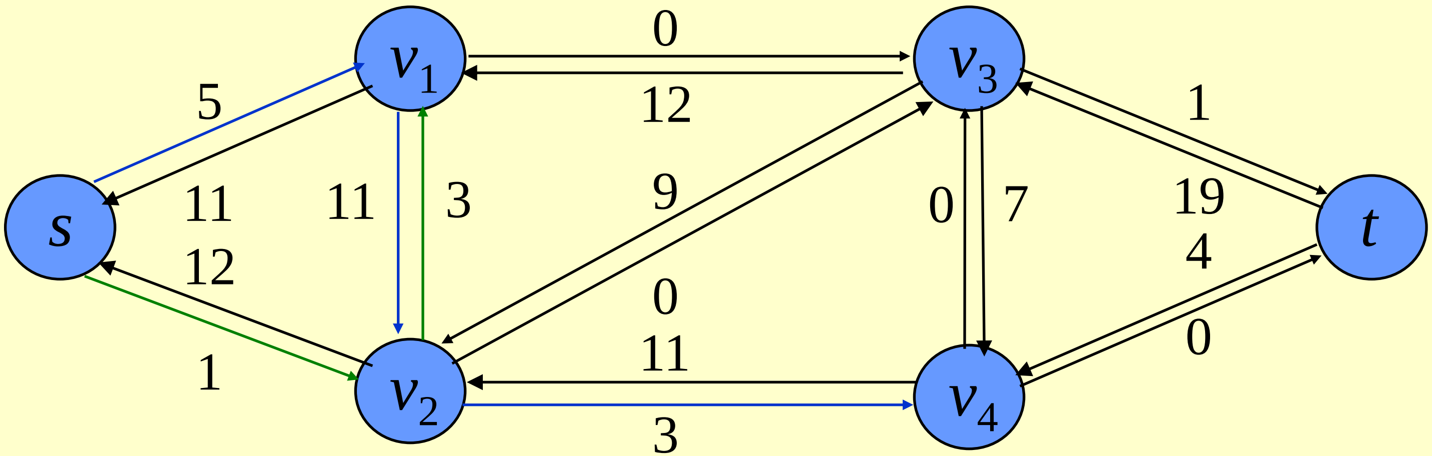


- Sample trace

Pushing a flow 4 on *p*4 (an augmenting path)



The corresponding residual network: no augmenting paths!



- Analysis of Ford-Fulkerson algorithm

  In practice, the maximum-flow problem often arises with integral capacities. If the capacities are rational numbers, an appropriate scaling transformation can be used to make them all integral. Under this assumption, a straightforward implementation of Ford-Fulkerson algorithm runs in time $O(E|f^*|)$, where f^* is the maximum flow found by the algorithm.

  The analysis is as follows:

  1. Lines 1-3 take time $\Theta(E)$.

  2. The while-loop of lines 4-8 is executed at most $|f^*|$ times since the flow value increases by at least one unit in each iteration. Each iteration takes $O(E)$ time.